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SPECTRUM ANALYSIS OF ROTOR BLADES BY MATRIX DISPLACEMENT METHOD AND MATRIX FORCE METHOD

J. H. Argyris, N. M. Lochner and D. W. Scharpf

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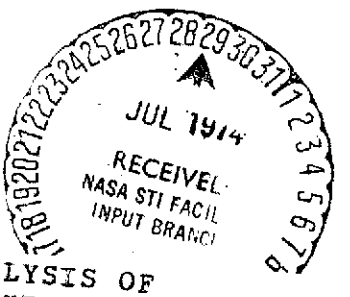

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16. Abstract The application of the matrix theory of statics and dynamics to the calculation of natural modes of oscillation of rotor blades is described. The basic principles of the matrix displacement method are summarized. The idealized rotor blade is broken down into elements; their elastic rigidities and kinematically equivalent point forces and moments applied to nodes, rather than being distributed, are derived. The corresponding kinematically equivalent mass matrix is also obtained. These and additional advanced techniques are shown to greatly facilitate rotor blade spectrum analysis. Matrix method results are compared with analytical and experimental results.

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Notation

Lightface	scalars
Boldface	matrices
^{t} (superscript)	transposed matrix
$^{-1}$ (superscript)	inverse matrix
Brackets	
[.....]	matrix
{.....}	column matrix
⌈.....⌋	diagonal matrix

SPECTRUM ANALYSIS OF ROTOR BLADES BY MATRIX DISPLACEMENT METHOD AND MATRIX FORCE METHOD

J. H. Argyris, N. M. Lochner and D. W. Scharpf

1. Introduction

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This report gives a description of how the matrix theory of statics and dynamics is applied to the calculation of natural modes of oscillation of rotor blades. Within this limited problem area, the authors have striven to provide the reader with an easily comprehended look at the new methods without assuming more than a knowledge of the elementary theory of bending and the basic concepts of matrix calculations.

A brief summary of the principles of the matrix displacement method is presented in the following section. Less value is placed here on providing proofs of theorems than on both testing theory and demonstrating its application through the use of selected examples. We see even in this section how well the matrix formulation of static and kinematic relations is suited to the requirements of large digital computers and the importance which thus must be assigned to the development of suitable software.

Starting from a simple consideration of the kinematic idealization of a rotor blade broken up into elements, both the elastic rigidities of the elements and the kinematically equivalent point forces and moments which must be applied at the nodes in place of a continuously or discontinuously distributed load are derived. Making logical use of the statically equivalent node forces, moreover, the corresponding kinematic equivalent mass matrix is also derived, which represents a great advance in the field of the dynamics of complex systems and has already been successfully applied to many tasks, particularly flutter calculations.

The same lines of reasoning result in a simplified representation of so-called geometric rigidity which has proven to be ideal for the computer treatment of problems with finite displacements, i.e. which strictly speaking are nonlinear. We use them in calculating the natural oscillations of rotor blades only to take the effect of centrifugal force into consideration, however.

In Fig. 4 we show how the required work can be decreased and the discussion can at the same time become more rigorous and clearer

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* Numbers in the margin indicate pagination in the foreign text.

if we use a mode of representation of deformations which is suitably adapted to the problem. This method, which provides for a breakdown into rigid-body displacements and so-called natural modes of deformation, is already facilitating the idealization of /136
rotor blades to a very great degree. Many relatively complex descriptions of wing assemblies can only be formulated reasonably in this manner.

The equations of motion of the rotor blade can now be written directly in the fifth section and converted into an eigenvalue problem for the dynamic matrix. The accuracy of the results obtained with this method is shown with the aid of an example involving comparison with analytically determined natural frequencies.

Of the many possibilities for representing a rotor blade in idealized form, we have presented several in Section 6 which the Institute's experience has shown to be particularly important.

While the above considerations are based exclusively on the matrix displacement method, the corresponding relations for the matrix force method are formulated in Section 7. Considering the duality of the method, it is possible to be very brief here. A special attempt is made to show that an essentially equivalent description of the properties of the rotor blade is possible.

Simple systems such as rotor blades can of course also in principle be calculated successfully with transfer matrices as described by Pestel and Marguerre [20].

The authors would like to express their acknowledgement of the valuable collaboration in preparation of the manuscript and the drawings provided by Miss Rosemarie Koch, Mrs. Erika Hofmann and Messrs. Wolfgang Kolmann and Jumbo Schneller, cand. aer.

2. Basic Principles of the Matrix Displacement Method[1] /137

2.1. Introduction

Let us first briefly recall several concepts in the matrix theory of statics [1, 2].

Stresses and strains in a volume element are written, in a three-dimensional state, as a 6 x 1 column matrix. The following notation proves to be particularly suitable in an oblique or cartesian system Oxyz:

[1] Abbreviated MDM

$$\sigma = \{\sigma_{xx} \quad \sigma_{yy} \quad \sigma_{zz} \quad \sqrt{2}\sigma_{xy} \quad \sqrt{2}\sigma_{yz} \quad \sqrt{2}\sigma_{zx}\} \tag{2.1}$$

and

$$\varepsilon = \{\varepsilon_{xx} \quad \varepsilon_{yy} \quad \varepsilon_{zz} \quad \frac{1}{\sqrt{2}}\varepsilon_{xy} \quad \frac{1}{\sqrt{2}}\varepsilon_{yz} \quad \frac{1}{\sqrt{2}}\varepsilon_{zx}\}. \tag{2.2}$$

For general anisotropy we write Hook's law in matrix form as

$$\sigma = \mathcal{C}\varepsilon , \tag{2.3}$$

where for an isotropic material in a system of cartesian axes, for example,

$$\mathcal{C} = 2G\left[I_6 + \frac{\nu}{1-2\nu}\begin{bmatrix} E_3 & O_3 \\ O_3 & O_3 \end{bmatrix}\right] \tag{2.4}$$

and G is sheer modulus, ν is Poisson's ratio, I_6 is a 6 x 6 unit matrix, O_3 is a 3 x 3 null matrix, and

$$E_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} . \tag{2.5}$$

The concepts of force and displacement are covered in general /138
terms in the theory. Thus a force can also refer to a moment or to a group of forces and moments. The same applies to displacement, which can refer to a displacement in the ordinary sense or to a rotation or a group of displacements or rotations.

2.2. Unit Displacement Law

The static relationships of the MDM are derived most elegantly with the unit displacement law[2], which in turn can be derived most simply from the principle of virtual displacements [2].

[2] Abbreviated UDL

Let some displacements $\mathbf{r}$ be given for a wing assembly. It should again be noted here that $\mathbf{r}$ can mean a group of displacements at different points. The problem posed is to calculate the force $\mathbf{R}$ which thereby occurs at a certain point or at a group of points in the wing assembly. We assume that the given displacements produce a mode of stress σ which is known. "Mode" means here that σ is a function of position which assigns a stress matrix to each point in accordance with equation (2.1). If $\mathbf{r}_1$ is a unit displacement vector and ε_1 is an associated, kinematically compatible (virtual) stress mode which produces

a) a unit displacement in the direction of the force R being sought at that point or those points at which R is applied, and

b) zero displacements for all forces acting upon the wing assembly except R,

then the UDL applies:

$$\boxed{\mathbf{r}_1^t \mathbf{R} = \int \varepsilon_1^t \sigma \, dV \ ,} \tag{2.6}$$

where we integrate over the entire volume of the wing assembly under consideration. Any desired measure can be used as "unit displacement vector" $\mathbf{r}_1$. If we seek only an individual force R, it is reasonable to introduce an actual unit displacement in its direction. In the latter case, we obtain the usual formulation of the UDL:

$$\boxed{1 \cdot R = \int \varepsilon_1^t \sigma \, dV .} \tag{2.7}$$

It should be observed that, as mentioned, ε_1 need satisfy the kinematic but not the static compatibility conditions; this can result in considerable simplifications. It is likewise sufficient that σ be only a stress mode which is statically compatible with R. Then σ may be in equilibrium with R but need not satisfy the kinematic compatibility conditions. We have introduced the underbar and overbar elsewhere to designate modes which satisfy only the kinematic or static compatibility conditions, respectively. Equation (2.7) can be generalized as follows with this convention:

$$\boxed{1 \cdot R = \int \underline{\varepsilon}_1^t \bar{\sigma} \, dV .} \tag{2.8}$$

2.3. Elastic Rigidity Matrix for a Wing Assembly Element

In the matrix theory of statics, a wing assembly is broken down into elements of finite size which are connected to one another at nodes. Each element is assigned a simplified stress mode which is unequivocally defined by the displacements at the associated node. We combine the node displacements to form a column matrix ρ_i for a typical element i. The number of elements in the matrix ρ_i is equal to the number of degrees of freedom of wing assembly element i which are taken into consideration in computation. Forces must either act upon the element at the nodes or, as we shall see later, be displaced to them in a kinematically compatible manner. For this purpose, we arrange these forces for element i in a column matrix P_i so that at corresponding positions in matrices ρ_i and P_i, displacements and forces exist, respectively, which refer to the same points and directions. The stresses ε_i at the i^{th} element are a linear function of displacement ρ_i which can be written in matrix form as

$$\varepsilon_i = \alpha_i \rho_i \,. \tag{2.9}$$

The matrix α_i thus contains the strain modes of element i resulting from unit displacements.

The elastic stresses are calculated from these strains, for example, using equation (2.3):

$$\sigma_i = E_i \varepsilon_i = E_i \alpha_i \rho_i \tag{2.10}$$

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In order to determine the loads corresponding to displacements ρ_i, we make use of the unit strain modes contained in matrix α_i in the UDL, equation (2.8), from which we obtain

$$P_i = \int \alpha_i^t E_i \alpha_i \rho_i \, dV = \left[\int \alpha_i^t E_i \alpha_i \, dV\right] \rho_i = k_{Ei} \rho_i \tag{2.11}$$

The matrix

$$k_{Ei} = \int \alpha_i^t E_i \alpha_i \, dV \tag{2.12}$$

is the elastic rigidity matrix for element i which we seek. The subscript E indicates that k_{Ei} has been obtained on the basis of

the elastic properties of the material and serves to distinguish this from geometric rigidity (see Section 3.3).

2.4. Rigidity of Overall Wing Assembly

The matrices of displacements ρ_i and forces P_i for all s elements of the wing assembly are collected in the column supermatrices

$$\varphi = \{\varphi_1 \; \varphi_2 \; \cdots \varphi_s\} \quad \text{and} \quad P = \{P_1 \; P_2 \; \cdots \; P_s\} \tag{2.13}$$

These matrices include all degrees of freedom of the elements that occur.

The displacements and the corresponding forces in the degrees of freedom actually to be taken into consideration in the given idealization at the nodes in the assembled wing section are represented by column matrices **r** and **R**. The matrix equation

$$\varphi = \mathbf{a}\mathbf{r}, \tag{2.14}$$

which is obtained through purely kinematic considerations, exists between matrices ρ and **r**. This kinematic relation can always be formulated, through a suitable choice of reference system, in such a manner that **a** is a Boolean matrix, i.e. one that contains only the numbers 0 and 1. Matrix **a** describes the connections between individual elements and also introduces the suspension conditions. /141

The kinematic relationship makes it possible to set up -- most easily with the aid of the UDL -- the equation

$$R = \mathfrak{a}^t P' \tag{2.15}$$

which describes static equilibrium.

If we collect the rigidities $\mathfrak{c}_i$ of the individual elements in the diagonal supermatrix

$$k_E = [k_{E1} \quad k_{E2} \quad \cdots \quad k_{En}] \qquad (2.16)$$

so that

$$P = k_E \varphi \qquad (2.17)$$

we can easily build up the overall rigidity K_E of the connected wing assembly. From the last equation we obtain, with equation (2.14),

$$P = k_E \varphi = k_E a r \qquad (2.18)$$

and (2.15)

$$R = a^t P = a^t k_E a r = K_E r \,, \qquad (2.19)$$

i.e.

$$K_E = a^t k_E a \qquad (2.20)$$

can be obtained by a congruent transformation of the matrix of element rigidities with the Boolean matrix **a** of kinematic relations.

For practical computation, it should be noted that it is desirable to subdivide matrices k_E and **a** into groups, e.g. on the basis of individual elements, and obtain K_E as the sum

$$K_E = \sum_i a_i^t k_{Ei} a_i \qquad (2.21)$$

since diagonal supermatrix k_E contains primarily zeros, which would take up an unnecessarily large amount of storage in the computer.

The SELMA [3] and BEMAT [4] programming languages, which contain standard routines for the treatment of Boolean matrices, were developed at our institute. Since only the coordinates of unity elements are stored for these matrices, very little storage space and computation time are required. /142

All calculations by the MDM can be automated to a very large degree, as has already been done by means of the ASKA programming system [5]. With commands which are extremely simple for the user, ASKA automatically calculates elemental and total rigidities and the displacements and stresses due to a given load from a minimum of data on geometry, materials properties, and the kinematics of a wing assembly, even for such large-scale problems that the core storage of the computer is no longer adequate by itself.

2.5. Example of a Homogeneous Beam Element with Uniaxial Flexure

As a simple example of the above theory, let us consider a homogeneous beam element with uniaxial flexure. The column matrices for displacements and forces are taken to be

$$\varphi = \{r_1 \mid r_2 \;\; \varphi_1 \;\; \varphi_2\} \quad \text{and} \quad P = \{S_1 \;\; S_2 \mid M_1 \;\; M_2\} \tag{2.22}$$

for an element[3] as shown in Fig. 2.1. The elastic curves v associated with the unit displacements are calculated on the basis of elementary flexure theory[4] and are plotted in Fig. 2.2. In EFT, if we neglect sheer deformation, we need consider only strain components ε_{zz}, so we easily find

$$\varepsilon = \frac{y}{l}\left[\frac{6\xi}{l} \mid -\frac{6\xi}{l} \quad (-1+3\xi) \quad (1+3\xi)\right]\varphi , \tag{2.23}$$

i.e. from Eq. (2.9),

$$\alpha = \frac{y}{l}\left[\frac{6\xi}{l} \mid -\frac{6\xi}{l} \quad (-1+3\xi) \quad (1+3\xi)\right] , \tag{2.24}$$

where y is distance from the neutral filament. We are already using the dimensionless coordinate

$$\xi = \frac{2z-l}{l} ,$$

---

[3] The subscript i used to designate element i is omitted in the following if not absolutely necessary.

[4] Abbreviated EFT

which is particularly desirable for describing the natural strain modes treated later. Integration of equation (2.12) yields /143

$$k_E = \frac{EI}{l^3} d\, p\, d \,, \tag{2.25}$$

where E = modulus of elasticity
I = areal moment of inertia
l = length of element

$$d = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$p = \begin{bmatrix} 12 & -12 & 6 & 6 \\ -12 & 12 & -6 & -6 \\ 6 & -6 & 4 & 2 \\ 6 & -6 & 2 & 4 \end{bmatrix} . \tag{2.26}$$

Using the system shown in Fig. 2.3 as an example, we shall now illustrate how total rigidity is built up. From equation (2.13), the column supermatrices are

$$q = \{ v_{11} \quad v_{21} \quad \varphi_{11} \quad \varphi_{21} \quad v_{12} \quad v_{22} \quad \varphi_{12} \quad \varphi_{22} \} \tag{2.27}$$

and

$$P = \{ S_{11} \quad S_{21} \quad M_{11} \quad M_{21} \quad S_{12} \quad S_{22} \quad M_{12} \quad M_{22} \} . \tag{2.28}$$

From Fig. 2.3, the column matrices for displacements and loads in the assembled system are

$$r = \{ \varphi_0 \quad v_1 \quad \varphi_1 \quad v_2 \quad \varphi_2 \} \qquad R = \{ M_0 \quad S_1 \quad M_1 \quad S_2 \quad M_2 \} . \tag{2.29}$$

With the aid of Fig. 2.3, we easily find the kinematic matrix a, shown in Table 2.1, which also includes total rigidity matrix K_E obtained from equation (2.21).

3. Kinematically Equivalent Mass Matrix and Geometric Rigidity /144

3.1. Kinematically Equivalent Idealization of Continuously Distributed Loads

As the previous section showed, application of the rigidity matrix requires that the load on an element be represented by fordes at its nodes. In general, however, there is a continuously distributed load on the system, such as

$$\mathcal{R} = \{U \quad V \quad W\}, \tag{3.1}$$

where U, V and W are the components of force per unit volume in the directions of the coordinate axes. We now wish to show how such a distributed load can be displaced to the nodes in a kinematically consistent manner.

The displacements of a point in this system are first collected in a column matrix $\mathcal{r}$. Corresponding to $\mathcal{R}$ from equation (3.1), for example, we must then set

$$\mathcal{r} = \{u \quad v \quad w\} \tag{3.2}$$

where u, v and w are displacements in the directions of the coordinate axes. For assumed modes of element deformation, $\mathcal{r}$ is a linear function of node displacements ρ, which can be expressed as

$$\mathcal{r}_i = \pi_i \rho_i \tag{3.3}$$

for a typical wing assembly element i. Unit displacement modes π_i must of course be kinematically compatible with unit strain modes α_i from equation (2.9), which was used to calculate the elastic rigidity matrix.

In order to determine the idealized forces P_i at the nodes of the elements corresponding to a distributed load $\mathcal{R}$, we use the UDL in the form

$$P_i = \int \pi_i^t \mathcal{R} \, dV \tag{3.4}$$

If we apply the rigidity matrix associated with the same mode of deformation, the node loads P_i calculated in this manner produce the same node displacements as the distributed load would.

3.2. Kinematically Equivalent Mass Matrix[5]

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Inertial forces are a particularly important case of the continuously distributed load, to which the aspects covered in the preceding section will now be applied [6, 7].

We write the relation between accelerations $\ddot{r}$ and inertial forces R_T in matrix form as

$$R_T = -m\ddot{r}. \tag{3.5}$$

If R and r are defined as in equation (3.1) and equation (3.2), for example, we must take matrix m to be

$$m = \int \rho \, dV \tag{3.6}$$

where δ indicates mass per unit volume, which can of course be a function of position.

To calculate the idealized inertial forces at the nodes, we now make consistent use again of the assumed modes of element deformation. Since the unit deformation modes in π_i are not functions of time, (3.3) indicates the accelerations of the i^{th} element to be

$$\ddot{r}_i = \pi_i \ddot{\varrho}_i \,, \tag{3.7}$$

from which, with equation (3.5), it follows that

$$R_{Ti} = -m_i \pi_i \ddot{\varrho}_i \,. \tag{3.8}$$

We now apply the UDL in the form of equation (3.4) and obtain

[5] Abbreviated KEM.

$$P_{ri} = -\int \pi_i^t m_i \pi_i \ddot{\varrho}_i dV = -\left[\int \pi_i^t m_i \pi_i dV\right]\ddot{\varrho} = -m_i \ddot{\varrho}_i \,. \quad (3.9)$$

The matrix

$$m_i = \int \pi_i^t m_i \pi_i dV \quad (3.10)$$

is the equivalent mass matrix for wing assembly element i, derived with kinematic consistency. It idealizes the arbitrarily distributed mass for each of the assumed element displacement modes by means of point and rotational masses at the nodes. In contrast to the normally employed idealization of distributed masses, the kinematically equivalent mass matrix of an element is not a diagonal matrix but, in general, is completely filled. As the examples in Section 3.4 show, the nodes are assigned equivalent rotational masses by the KEM for a beam element, even though rotational inertia is neglected. The KEM for a complete wing assembly, like total rigidity, is obtained by the congruent transformation

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$$M = a^t m a \,, \quad (3.11)$$

where

$$m = [m_1 \quad m_2 \quad \cdots \quad m_s] \quad (3.12)$$

is the diagonal supermatrix of KEMs for all elements.

3.3. Geometric Rigidity

To extend the MDM to large displacements, the method of geometric rigidity has been developed such as described in [7, 8, 10]. At this point we wish to limit ourselves to explaining that material necessary for understanding the following[6].

In the undeformed rotor blade, centrifugal forces generate only displacements in the longitudinal direction of the blade, which need not be taken into consideration in calculating natural oscillations. Normal force becomes operant only on the deformed blade. If we designate the state of displacement as $\vec{w}$ here,

---

[6] See also Section 4.4.

too, which can be defined for a rotor blade in accordance with Fig. 3.2, for example, then we take the forces which occur as a result of the change in the geometric position of a longitudinal element of the blade to be -- in the first approximation -- a linear function

$$\mathfrak{R}_0 = q r \,. \tag{3.13}$$

The matrix q can always be easily determined, as the example in Section 3.4 shows. As we have done already in deriving the KEM, we determine the node loads associated with the assumed displacement modes which the continuously distributed load $\mathfrak{R}_0$ produces to be

$$P_i = \int \pi_i^t \mathfrak{R}_G dV = \int \pi_i^t q_i r_i dV = \left[\int \pi_i^t q_i \pi_i dV\right] \rho_i = k_{Gi} \rho_i \tag{3.14}$$ /147

for a typical blade element, with the aid of the UDL, and we call the matrix

$$k_{Gi} = \int \pi_i^t q_i \pi_i dV \tag{3.15}$$

the geometric rigidity of the wing assembly element.

For a given displacement ρ_i of the i^{th} wing assembly element, the total load at its nodes becomes

$$P_i = k_{Ei}\rho_i + k_{Gi}\rho_i = (k_{Ei} + k_{Gi})\rho_i = k_i \rho_i \,, \tag{3.16}$$

elastic geometric

i.e., the total rigidity of the element is obtained simply as the sum

$$k_i = k_{Ei} + k_{Gi} \tag{3.17}$$

It is apparent that the rigidity of the entire system is built up just as done in equation (2.20).

3.4. Examples

a) Kinematically Equivalent Idealization of a Triangular Load on a Beam Element

Fig. 3.1 shows a beam element with a so-called triangular load

$$R(\xi) = \frac{1}{2} q_0 (1 - \xi) , \tag{3.18}$$

which will now be shifted to forces at the nodes. We utilize the deformation modes in Fig. 2.2. We need only displacement v in matrix $\mathfrak{N}$, and thus easily find the matrix defined in equation (3.3)

$$\pi = \left[\frac{1}{4}(2 - 3\xi + \xi^3) \quad \frac{1}{4}(2 + 3\xi - \xi^3) \quad \frac{1}{8}(1 - \xi - \xi^2 + \xi^3) \quad \frac{1}{8}(-1 - \xi + \xi^2 + \xi^3) \right] . \tag{3.19}$$

Integration of equation (3.4) yields /148

$$P = q_0 l \left\{ \frac{7}{20} \quad \frac{3}{30} \quad \frac{1}{20} \quad -\frac{1}{30} \right\} .$$

We use these idealized forces to make calculations for a cantilever beam (see Fig. 3.1), which is represented by one element. The rigidity matrix associated with the assumed displacement modes has already been given in equation (2.25). The reader can easily verify that we thus obtain

$$r = \frac{q_0 l^4}{30 E I} \qquad \varphi = \frac{q_0 l^4}{24 E I}$$

for the displacements. Thus although the actual elastic curve cannot be obtained by a linear combination of the assumed displacement modes, we arrive at the exact results, as was to be shown here.

b) Kinematically Equivalent Mass Matrix for a Homogeneous Beam Element

In calculating the KEM for the homogeneous beam element, we want to take not only displacement v but also rotation ϕ in the cross section into consideration. Fig. 3.2 shows how v and ϕ are defined. The quantities s' and m' are the force and moment per unit length, respectively. The matrix π in equation (3.19) must now be expanded to

$$\pi = \frac{1}{8}\left[\begin{array}{cc|cc} 2(2-3\xi+\xi^3) & & 2(2+3\xi-\xi^3) & 1(1-\xi-\xi^2+\xi^3) \quad 1(-1-\xi+\xi^2+\xi^3) \\ \frac{12}{l}(-1+\xi^2) & & \frac{12}{l}(1-\xi^2) & 2(-1-2\xi+3\xi^2) \quad 2(-1+2\xi+3\xi^2) \end{array}\right] \tag{3.20}$$

for angle of rotation ϕ. For the inertial forces per length element we have

$$s_T' = \mu \ddot{v} \quad \text{and} \quad m_T' = \mu i^2 \ddot{\varphi} \,, \tag{3.21}$$

where μ = mass per unit length
i = radius of gyration of cross section,
from which we obtain

$$m = \mu \left[\begin{array}{cc} 1 & i^2 \end{array}\right] . \tag{3.22}$$

For the KEM, calculation of the integral in equation (3.10) yields /149

$$m = \mu d \left[\frac{l}{420} m_1 + \frac{i^2}{30\,l} m_2\right] d \tag{3.23}$$

where

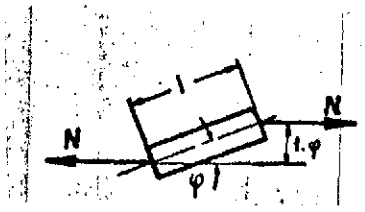
$$d = \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \end{array}\right]$$

with the purely numerical matrices

$$m_1 = \begin{bmatrix} 156 & 54 & 22 & -13 \\ & 156 & 13 & -22 \\ & & 4 & -3 \\ \text{symm.} & & & 4 \end{bmatrix} \quad \text{and} \quad m_2 = \begin{bmatrix} 36 & -36 & 3 & 3 \\ & 36 & -3 & -3 \\ & & 4 & -1 \\ \text{symm.} & & & 4 \end{bmatrix}$$

c) Geometric Rigidity of a Homogeneous Beam Element

As a demonstration, we now calculate the geometric rigidity of a homogeneous beam element with constant normal force N. As can be easily seen from Fig. 3.3, force N generates only a bending moment



$$m_0' = N\varphi \tag{3.24}$$

Fig. 3.3

Thus we can write

$$q = \begin{bmatrix} 0 & N \end{bmatrix} \tag{3.25}$$

and, from (3.15),

$$k_0 = \frac{N}{30\,l} d\, m_2 d \,. \tag{3.26}$$

It is apparent that it must be possible to obtain k_G simply from the portion of the KEM which takes rotation in the cross sections into consideration if we just replace μi^2 with N.

4. The Method of Natural Deformation Modes[7] /150

4.1. Introduction

The calculation of rigidity and mass matrices is considerably simplified conceptually, too, if we break down the displacements

[7] Described in detail in [7, 9, 10].

of wing assembly elements into

Rigid body displacements ρ_O

and

Stress-generating deformations ρ_N, which we call natural,

as is done later, for example, in Fig. 4.1 and 4.2 for the beam element. With the aid of easily determined transformation matrices a_N and a_O, we obtain

$$\rho_N = a_N \rho \tag{4.1}$$

and

$$\rho_O = a_O \rho . \tag{4.2}$$

These equations are collected to form

$$\rho' = a_e \rho , \tag{4.3}$$

where

$$\rho' = \{\rho_N \quad \rho_O\} \tag{4.4}$$

and

$$a_e = \{a_N \quad a_O\} . \tag{4.5}$$

Load matrices P_N and P_O are defined to correspond to ρ_N and ρ_O; these are likewise drawn in in Figs. 4.1 and 4.2 for the beam element. We also collect these load vectors to form

$$P' = \{P_N \quad P_O\} . \tag{4.6}$$

On the basis of the principle of virtual displacements, we obtain the dual relationship to equation (4.2) /151

$$P = a_e^t P' = a_N^t P_N + a_O^t P_O . \tag{4.7}$$

Observe that the matrices a_N, a_O can be derived from both kinematic and static considerations.

4.2. Elastic Rigidity Matrix

It is apparent that only the natural deformations need be taken into consideration for calculating the elastic rigidity matrix. Equation (2.9) then becomes

$$\varepsilon = a_N \varrho_N \,, \tag{4.8}$$

where a_N retains only strain modes due to natural unit displacements. We shall see in the following examples that calculations for a beam element are thus already highly simplified. Complex two- and three-dimensional elements can in practice be formulated reasonably only through the application of this methodology. From equation (2.12), the natural elastic rigidity matrix becomes

$$k_{NE} = \int a_N^t \mathfrak{E} a_N \, dV \,. \tag{4.9}$$

Since the resultant of elastic forces is of course zero, it follows from equations (4.1) and (4.7) that

$$P = a_N^t P_N = a_N^t k_{NE} \varrho_N = a_N^t k_{NE} a_N \varrho \,, \tag{4.10}$$

i.e. we obtain the following for the elastic rigidity matrix of the wing assembly element:

$$k_E = a_N^t k_{NE} a_N \tag{4.11}$$

It should also be noted that in individual cases, e.g. for bar-shaped wing assemblies, we can usually determine natural compliance f_{NE} more easily, and from this calculate

$$k_{NE} = f_{NE}^{-1} \tag{4.11a}$$ /152

4.3. Kinematically Equivalent Mass Matrix

Calculation of the KEM also leads, as the following examples will show, to considerably simpler formulations if the method of natural displacements is applied. The displacement v of a point in the element is now determined using

$$v = \alpha_N \rho_N + \alpha_0 \rho_0 = \alpha_e \rho' \tag{4.12}$$

where

$$\alpha_e = \begin{bmatrix} \alpha_N & \alpha_0 \end{bmatrix} \tag{4.13}$$

The matrices α_N and α_0 thus contain displacements due to natural or rigid unit displacements. With simpler integrals, we now calculate the modified KEM in place of equation (3.10):

$$m' = \int \alpha_e^t m \alpha_e \, dV \tag{4.14}$$

and for the normal KEM, using (4.3) and (4.7), we obtain

$$m = a_e^t m' a_e \tag{4.15}$$

4.4. Geometric Rigidity

Application of the method of natural deformation loads leads to a deeper understanding of the nature of geometric rigidity [7, 8, 10]. We merely wish to mention here that geometric rigidity generally is a function, as a first approximation, of natural load P_N and the running geometry, but not of the content of the element. For example, the geometric rigidity of a triangle formed by three bars is, as a first approximation, the same as that of a corresponding triangular membrane element. It should only be noted here that k_G can be calculated with considerably less effort, as in Section 4.3., via modified geometric rigidity /153

$$k_G' = \int \alpha_e^t g \alpha_e \, dV \tag{4.16}$$

and

$$k_G = a_e^t k_G^i a_e \tag{4.17}$$

where a_e and α_e are again defined by equations (4.5) and (4.13).

4.5. Examples

We wish to repeat application of the method of natural deformations to a beam element from [10].

Fig. 4.1 shows how desirably the natural deformation modes are selected as symmetric and antisymmetric forms of flexure ϕ_S and ϕ_A, respectively. Accordingly, we take the symmetric and antisymmetric moments M_S and M_A, respectively, as natural loads. The displacement modes for the element treated as a rigid body are described as in Fig. 4.2 with translation r_T and rotation r_R. We designate the corresponding loads, likewise shown there, as R_T and R_R.

If we again base the displacement loads of the beam element on those in Fig. 2.2, they can be broken down into natural deformations and rigid body displacements as shown in Fig. 4.3. We thus easily find the transformation matrices

$$a_N = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 2/l & -2/l & 1 & 1 \end{bmatrix} \tag{4.18}$$

and

$$a_0 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}. \tag{4.19}$$

Just as easily, we confirm that, for $r = \{v \ \varphi\}$ (Fig. 3.2),

$$\alpha_N = \begin{bmatrix} \frac{1}{8}(1-\xi^2) & -\frac{1}{8}\xi(1-\xi^2) \\ -\xi/2 & -\frac{1}{4}(1-3\xi^2) \end{bmatrix} \tag{4.20}$$

and

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$$a_0 = \begin{bmatrix} 1/2 & 1/2\,\zeta \\ 0 & 1/1 \end{bmatrix} , \tag{4.21}$$

where ζ is again the dimensionless coordinate already explained.

The elastic rigidity of the element is now first calculated via natural elastic compliance. From the definition of natural load, we obtain the section moment here:

$$M = m_N P_N \tag{4.22}$$

where

$$m_N = \begin{bmatrix} 1 & -\zeta \end{bmatrix} . \tag{4.23}$$

As will be shown in the section on the matrix force method, equation (7.15), we thus obtain

$$f_{NE} = \frac{l}{2EI} \int_{-1}^{+1} m_N^t m_N \, d\zeta , \tag{4.24}$$

where $\frac{l}{2} d\zeta = dz$

E = modulus of elasticity

I = moment of inertia of cross-sectional area

Now only two extremely simple integrals remain to be determined in order to obtain

$$f_{NE} = \frac{l}{EI} \begin{bmatrix} 1 & \frac{1}{3} \end{bmatrix} \tag{4.25}$$

and thus

$$k_{NE} = f_{NE}^{-1} = \frac{EI}{l} \begin{bmatrix} 1 & 3 \end{bmatrix} \tag{4.26}$$

The reader can easily verify that equation (4.11), with a_N from equation (4.18), actually yields the elastic rigidity matrix which was already given in (2.25). Naturally, this congruent transformation will only be carried out on computers in more complex cases.

As a comparison of matrices ϖ_N and ϖ_Q from equations (4.20) and (4.21), respectively, with matrix π from equation (3.20) shows, considerably simpler integrals must also be determined in the calculation of matrices m' and k_G' from equations (4.14) and (4.16), respectively. In particular, we immediately see that most /155
of the integrals yield zero due to the orthogonality of the functions contained in ϖ_N and ϖ_Q . The reader can easily verify that the following thus hold:

modified geometric rigidity

$$k_G' = \frac{N}{60l} d' p_G' d' \tag{4.27}$$

and modified KEM

$$m' = d'\left(\frac{\rho l}{840} m_1' + \frac{\rho i^2}{60l} m_2'\right) d' \tag{4.28}$$

where

$$d' = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

$$p_G' = m_2' = \begin{bmatrix} 5 & 3 & 0 & 60 \end{bmatrix}$$

$$m_1' = \begin{bmatrix} 7 & 0 & 35 & 0 \\ & 1 & 0 & -7 \\ & & 210 & 0 \\ \text{symm.} & & & 70 \end{bmatrix}.$$

5. Calculation of Natural Oscillations /156

5.1. The Dynamic Matrix

The equations of motion for natural oscillations of the system are obtained by D'Alembert's principle in the MDM as

$$M\ddot{r} + Kr = 0. \tag{5.1}$$

This system of coupled linear differential equations is solved with the expression

$$r = r_0 e^{i\omega t} \tag{5.2}$$

with which we find

$$(\omega^2 M - K) r_0 = 0 \tag{5.3}$$

We can assume here that mass matrix M for a rotor blade is not singular[8], and thus transform equation (5.3) into

$$(M^{-1}K - \omega^2 I) r_0 = 0 , \tag{5.4}$$

where I is the unit matrix. We define

$$D = M^{-1}K \tag{5.5}$$

as the dynamic matrix for the system. Thus equation (5.4) represents an eigenvalue problem for dynamic matrix D. From eigenvalues of D we directly obtain the natural frequencies, and the eigenvectors of D yield the natural modes of oscillation. Many proven methods exist for solving matrix eigenvalue problems, of which we particularly wish to emphasize the Aronszajn-Patton algorithm [12, 13], which has also proven itself quite well for extremely difficult aeroelastic problems. From the eigenvectors r_{0i} we obtain the associated loads on the wing assembly elements with

$$P_i = k a r_{0i} . \tag{5.6}$$

5.2. Examples

a) Cantilever Beam with One Element:

As the simplest possible example, we show a cantilever beam in Fig. 5.1 with a homogeneous mass distribution represented by one element. From kinematic matrix a (see Fig. 5.1), and matrices k_E, equation (2.25) and m, equation (3.23), it becomes apparent that

[8] More general formulation in [11].

$$K_e = \frac{EI}{l^3}\begin{bmatrix} 12 & -6l \\ -6l & 4l^2 \end{bmatrix} \quad \text{and} \quad M = \frac{\mu l}{420}\begin{bmatrix} 156 & -22l \\ -22l & 4l^2 \end{bmatrix}$$

from which

$$D = \frac{3EI}{\mu l^4}\begin{bmatrix} -84 & 64 \\ -672 & 492 \end{bmatrix}$$

follows from equation (5.5) for the dynamic matrix. The fundamental frequency of this cantilever beam is found from the smallest eigenvalue of to be

$$\nu_{MDM} = 0.5621\sqrt{\frac{EI}{\mu l^4}}$$

in place of the value, calculated from the EFT, of

$$\nu_{EFT} = 0.5595\sqrt{\frac{EI}{\mu l^4}} .$$

It should also be noted that the calculation of buckling load using geometric rigidity results in a similar eigenvalue problem [10].

b) Rotatably Mounted Bending Beam with Constant Normal Force:

Table 5.1 shows the natural oscillations of a system similar to a rotor blade as calculated by the MDM. The rigidity and mass matrices in equations (2.25), (3.23) and (3.26) have been used for this homogeneous bending beam mounted free to rotate. This simple problem can also be easily solved analytically, so the accuracy of the MDM results can be checked. It is particularly notable how well even the fifth natural frequency agrees with the analytical value for just five elements.

6. Idealization of Rotor Blade Elements /158

6.1. Inhomogeneous Blade Element with Uniaxial Flexure

As already mentioned in the theoretical derivation of the rigidity and mass matrices, any distribution of flexural rigidity or

compliance, of mass and of normal force within the element can be taken into consideration. We wish to cover the case of a linear function here, particularly important in practice. Let the following be given:

	at left end of element	at right
Flexural rigidity	EI_1	EI_2
Mass per unit length	μ_1	μ_2
Normal force	N_1	N_2

We assume that μ and N vary linearly between the values at the ends. For the sake of simplicity, we take flexural compliance

$$\phi = \frac{1}{EI}$$

to be a linear function.

The normal forces at the ends of each element can easily be calculated exactly from the centrifugal forces due to the given mass distribution and angular velocity Ω of the blade. According to our experience, however, it is then sufficient to take a linear variation of the normal force within the element into consideration.

We now introduce (see also [14]) the means

$$\phi_M = \frac{1}{2}(\phi_1 + \phi_2) \qquad \mu_M = \frac{1}{2}(\mu_1 + \mu_2) \qquad N_M = \frac{1}{2}(N_1 + N_2)$$

and the differences

$$\phi_D = \frac{1}{2}(\phi_1 - \phi_2) \qquad \mu_D = \frac{1}{2}(\mu_1 + \mu_2) \qquad N_D = \frac{1}{2}(N_1 + N_2),$$

since the corresponding matrices of the homogeneous element can then be used for the means. We also use the abbreviation

$$\varkappa = \frac{\phi_D}{\phi_M}$$

We thus obtain the following for the elastic rigidity matrix /159

$$k_e = \frac{1}{l^3 \phi_M (1 - \frac{1}{3}\varkappa^2)} d (R_{e1} + 2\varkappa R_{e2}) d, \tag{6.1}$$

the geometric rigidity matrix

$$k_g = \frac{1}{30l} d\left[N_M(P_{G1} + 2xP_{G2}) + N_S(P_{G3} + \frac{2}{7}xP_{G4})\right]d \qquad (6.2)$$

and the KEM

$$m = \frac{1}{420} d\left[\mu_M(m_1 + \frac{1}{2}xm_2) + \mu_S m_3\right]d, \qquad (6.3)$$

where

$$d = \left[1 \quad l \quad l \quad l\right]. \qquad (6.4)$$

The numerical matrices are all given in Table 6.1 for the sake of completeness, although P_{G1} and P_{01} and m_1 are of course already known. Since they are unimportant for calculating rotor blade oscillations, we have neglected the following: in equation (6.2), the effect of rotational activity μi^2; in equations (6.2) and (6.3), the small second-order quantities x^2 and $x\left(\frac{\mu_S}{\mu_M}\right)$. The natural frequencies of a tapered rotor blade are shown in Table 6.2. These results show how rapidly the natural frequencies converge with the number of elements used if we already take inhomogeneity into consideration in the elements. Only the actual bending oscillations are covered in the table.

6.2. Homogeneous Blade Element with Shear Deformation /160

Shear deformation can be covered in two ways in the MDM; these have already been treated [15]. The displacements of nodes due to shear can be treated either as additional independent variables or in conjunction with displacements due to flexure. In pure beam designs, it is apparently best to use only the sum of both displacements, particularly because separate shear deformations result in poorly "conditioned" rigidity matrices. If beam elements are connected with additional structural components (e.g. a skin), however, it is necessary to consider separate shear and bending deformations. The result of this is that very subtle lines of reasoning are required in dynamic calculations in order to solve these problems [16]. All of these difficulties are circumvented by the SPARTA spar element [17], in whose eight nodes only one v displacement is possible in each case.

We first treat the method in which additional degrees of freedom are introduced for displacements due to shear. We therefore expand the matrices for node displacements and loads, equation (2.22):

$$q = \{r_1 \quad r_2 \quad \varphi_1 \mid \varphi_2 \quad s_1 \quad s_2\} \tag{6.5a}$$

and

$$P = \{S_1 \quad S_2 \quad M_1 \mid M_2 \quad S_1 \quad S_2\}. \tag{6.5b}$$

Fig. 6.1 shows the displacement modes associated with the new degrees of freedom s_1 and s_2, which are added to those in Fig. 2.2. It is apparent that "shear displacements" s_1 and s_2 correspond to the same forces as "flexural displacements" r_1 and r_2, resulting in equation (6.5b).

It must be especially emphasized that ϕ_1 and ϕ_2 at the ends of the element, like ϕ within it, now represent only rotation of the cross section and no longer represent the tangent to the elastic curve at the same time, which we designate v'. Pure shear deformation does indeed produce a v', but no ϕ. We can actually write

$$v' = \phi + \gamma, \tag{6.6}$$

where γ is angle of shear.

For inertial forces on the longitudinal element we now have /161

$$s_T' = \mu \ddot{v} \quad \text{and} \quad m_T' = \mu i^2 \ddot{\varphi}; \tag{6.7}$$

for moment due to normal force, however,

$$m_G' = Nv'. \tag{6.8}$$

We must thus set the matrices $\varkappa$ for displacement of the longitudinal element as follows:

for geometric rigidity

$$\varkappa_G = \{v \quad v'\} \tag{6.9a}$$

and

for the kinematically consistent mass matrix

$$\varkappa_M = \{v \quad \varphi\} \tag{6.9b}$$

Further derivation of the 6 x 6 rigidity matrices can be passed over following these explanations. We merely show the results in Table 6.3.

We now use

$$v_1 = r_1 + s_1 \text{ and } v_2 = r_2 + s_2 \tag{6.10}$$

to designate the total displacement of the left and right nodes, respectively, so

$$q = \{v_1 \mid v_2 \quad \varphi_1 \quad \varphi_2\} \tag{6.11}$$

holds, while of course

$$P = \{S_1 \mid S_2 \quad M_1 \quad M_2\}$$

again, as in equation (2.22).

We can derive the rigidity and mass matrices very elegantly if we make use of the natural deformation modes. Let us begin, as in Section 4.5, with the natural compliance of the beam element.

Again equation (4.23) applies to section moments due to natural unit forces, and for section shear forces we easily find /162

$$\gamma_N = -\frac{2}{l}\begin{bmatrix} 0 & 1 \end{bmatrix}, \tag{6.12}$$

where it is apparent that a shear force is associated only with antisymmetric deformation. We can now expand equation (4.24) to shear deformation

$$f_{NE} = \frac{2}{l}\left[\frac{1}{EI}\int_{-1}^{+1} m_N^t m_N \, d\xi + \frac{1}{GC}\int_{-1}^{+1} \gamma_N^t \gamma_N \, d\xi\right] \tag{6.13}$$

where G is the modulus of shear and C is effective shear cross section. Simple integration of the last equation yields

$$f_{NE} = \frac{l}{EI}\begin{bmatrix} 1 & \frac{1}{3}(1+\alpha) \end{bmatrix}, \qquad (6.14)$$

where

$$\alpha = \frac{12EI}{l^2GC}. \qquad (6.15)$$

Since the transformation matrices a_N and a_O are of course not affected by shear deformation, we can take them from equations (4.18) and (4.19), respectively, and again calculate k_E with equations (4.11) and (4.11a).

In calculating geometric rigidity and the KEM, too, a change occurs only in the antisymmetric flexure case, as opposed to the derivations in Section 4.5. We immediately see from a comparison of the natural elastic compliances from equations (4.25) and (6.14) that the rotations ϕ_1 and ϕ_2 at the ends of the beam element for $\phi_A = 1$ (Fig. 4.1) are reached under the action of an asymmetric moment which is smaller by a factor of $1/(1 + \alpha)$ if shear deformation is taken into consideration. It follows from this that the tangent slopes at the end points must be

$$v_1' = v_2' = \frac{1}{2(1+\alpha)} \qquad (6.16)$$

and, from equation (6.8), we thus find the shear angle for natural deformation $\phi_A = 1$ to be

$$\gamma = \frac{1}{2(1+\alpha)} - \frac{1}{2} = -\frac{\alpha}{2(1+\alpha)}. \qquad (6.17)$$

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We can easily see on the basis of these considerations that

$$\kappa_G = \omega_0 S \varphi' \qquad (6.18)$$

for calculating geometric rigidity (see quation (6.9a)), where ω_0 is built up on the basis of equation (4.13) with the transformation matrices from equations (4.20) and (4.21) for the beam element without shear deformation, and

$$S = \begin{bmatrix} 1 & \frac{1}{1+\alpha} & 1 & 1 \end{bmatrix} \qquad (6.19)$$

is a simple diagonal matrix. We can take S from the integrals in equation (4.16) and thus find modified geometric rigidity with shear stress to be

$$k_{GS}' = S\, k_G' S \quad , \tag{6.20}$$

where k_G' is given in equation (4.27).

In calculating the KEM, it is of course necessary to consider the angle of shear, which produces an additional rotation of the cross section in the antisymmetric natural deformation mode. In matrix form, we thus write

$$\varkappa_M = \left[\varpi_e + \varpi_T\right] S q' \tag{6.21}$$

for $\varkappa_M$ from equation (6.9b), where

$$\varpi_T = \frac{\alpha}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{6.22}$$

can easily be found from equation (6.17). For the matrix term ϖ_T it is thus still necessary to determine several trivially simple integrals; the modified KEM, taking shear deformation into consideration, is

$$m_S' = S\left[m' + \frac{\mu i^2}{l} d' m_T d'\right] S \tag{6.23}$$

where m' (equation (4.28)) and S are already known, and

$$m_T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ & \alpha^2/4 & 0 & \alpha/2 \\ & & 0 & 0 \\ \text{symmet.} & & & 0 \end{bmatrix}$$

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If we compare our formulation of these rigidity and mass matrices

with those by Uhrig [18], we see quite clearly what kind of simplification the method of natural deformation modes can offer.[9]

As a practical application of the method described here, the natural frequencies of a beam are given in Table 6.4 for which shear deformation and rotational inertia must be taken into consideration.

6.3. Twisted Blade Element with Coupling between Flapping and Azimuthal Flexure and Torsion

Using the method described in Section 6.2, we now perform calculations for a twisted rotor blade element, in which biaxial flexure must of course be assumed. In order to give our representation a relatively simple form, we limit ourselves here to the case in which twisting occurs about an axis through the centroids $\hat{G}$. The effect of shear deformation and the coupling between flexure and torsion are taken into consideration here.

We describe the elastic properties, first of all, by means of the flexural compliances referred to the principle axes $\hat{G}\hat{x}$ and $G\hat{y}$ of each cross section, collected in a diagonal matrix

$$\Phi = \begin{bmatrix} \Phi_I & \\ & \Phi_{II} \end{bmatrix} = \begin{bmatrix} \frac{1}{EI_\eta} & \\ & \frac{1}{EI_\xi} \end{bmatrix} \tag{6.24}$$

and, secondly, by means of the 2 x 2 shear compliance matrix

$$\Psi = \begin{bmatrix} \Psi_{II} & \Psi_{I\,II} \\ \Psi_{II\,I} & \Psi_{II\,II} \end{bmatrix} = \begin{bmatrix} \frac{1}{GC_{II}} & \frac{1}{GC_{I\,II}} \\ \frac{1}{GC_{II\,I}} & \frac{1}{GC_{II\,II}} \end{bmatrix} \tag{6.25}$$

Here, I,II indicate the two principal axial directions. In explanation of equation (6.25), it is noted that, in general, coupling of shear deformations is also known to occur during flexure in a principal axis plane [19]. /165

The angle of rotation β between the spatially fixed coordinate axis x and the principal axis $\hat{x}$ (see Fig. 6.2) is assumed to be given by

$$\beta = \beta_M + \beta_D \zeta \tag{6.26}$$

[9] Uhrig even neglects the terms with α^2 there. In the case of finer subdivision, as is particularly necessary for calculating higher natural frequencies, we obtain unusable results because of this omission.

where β_M is the angle between x and $\hat{x}$ at the center of the element, and ζ is the dimensionless coordinate already used several times. The twist within an element, i.e. β_D from equation (6.26), is assumed to be small.

For the column matrices of displacements and forces, respectively, at nodes in this element, for which 10 degrees of freedom must be taken into consideration, we define

$$\mathbf{q} = \left\{ v_{1y} \quad v_{2y} \quad \varphi_{1x} \quad \varphi_{2x} \quad v_{1x} \quad v_{2x} \quad \varphi_{1y} \quad \varphi_{2y} \quad \vartheta_1 \quad \vartheta_2 \right\} \tag{6.27a}$$

and

$$\mathbf{p} = \left\{ S_{1y} \quad S_{2y} \quad M_{1x} \quad M_{2x} \quad S_{1x} \quad S_{2x} \quad M_{1y} \quad M_{2y} \quad T_1 \quad T_2 \right\} \tag{6.27b}$$

As Fig. 6.3 shows, displacements v and angles of rotation ϕ, as well as the corresponding forces S and moments M, are referred to the directions of the spatially fixed x and y coordinate axes; the angles θ and moments T of torsion, to the z-axis passing through the centroids.

We select the known symmetric and antisymmetric bending modes (Fig. 4.1), which are taken for the y-z and x-z planes in each case, as natural displacement modes of the element due to flexure and shear. For natural displacement due to torsion it is desirable to use

$$\vartheta_N = \vartheta_2 - \vartheta_1 \tag{6.28}$$

This angle θ_N does not refer to a torsion about the axis of centroids G, like θ_1 and θ_2, but to the elastic axis E_S, which in general does not coincide with the axis of centroids. This makes it possible to select the natural deformation modes in such a manner that no coupling occurs between flexure and torsion. Thus for the natural displacement vector we set

$$\mathbf{q}_N = \left\{ \varphi_{Sx} \quad \varphi_{Ax} \quad \varphi_{Sy} \quad \varphi_{Ay} \quad \vartheta_N \right\} = \left\{ \mathbf{q}_{NBs} \;\middle|\; \mathbf{q}_{NT} \right\} \tag{6.29}$$

where

$$\mathbf{q}_{NBs} = \left\{ \varphi_{Sx} \quad \varphi_{Ax} \quad \varphi_{Sy} \quad \varphi_{Ay} \right\} \qquad \mathbf{q}_{NT} = \vartheta_N$$

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and, correspondingly, for the natural load vector

$$P_N = \{ M_{Sx} \quad M_{Ax} \quad M_{Sy} \quad M_{Ay} \quad T_N \} = \{ P_{NBs} \quad P_{NT} \} \tag{6.30}$$

where

$$P_{NBs} = \{ M_{Sx} \quad M_{Ax} \quad M_{Sy} \quad M_{Ay} \} \qquad P_{NT} = T_N .$$

Table 6.5 shows how transformation matrix a_N is built up. It is apparent that the matrix from equation (4.18) can be taken for the flexure and shear portions. In Fig. 6.4, the displacement mode $\theta_2 = 1$ is used as an example to illustrate that torsions θ_1 and θ_2 about $\hat{G}$ also generate natural flexural deformations through the displacement of the reference point of natural torsion θ_N. Thus the coupling between flexure and torsion is introduced via the congruent transformation in equation (4.11).

We now first cover natural elastic compliance without torsion. It is apparent that the section moments and section forces due to natural unit forces are

$$m_N = \begin{bmatrix} 1 & -f & 0 & 0 \\ 0 & 0 & 1 & -f \end{bmatrix} \tag{6.31}$$

and

$$q_N = -\frac{2}{l} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \tag{6.32}$$

We transform the natural forces to the principal axis directions in the center of the element with the matrix /167

$$L_{Me} = \begin{bmatrix} \cos\beta_N I_2 & \sin\beta_N I_2 \\ -\sin\beta_N I_2 & \cos\beta_N I_2 \end{bmatrix} . \tag{6.33}$$

Equations (6.31) and (6.32) also apply to these new axes. The resultant section moments and transverse forces are transformed to the principal axes of each cross section by

$$\mathcal{L}_D = \begin{bmatrix} \cos\beta_D\xi & \sin\beta_D\xi \\ -\sin\beta_D\xi & \cos\beta_D\xi \end{bmatrix} \tag{6.34}$$

where -- since β_D is assumed to be small -- it is sufficient to consider terms of up to second order, so

$$\mathcal{L}_D = (1 - \tfrac{1}{2}\beta_D^2\xi^2)\mathbf{I}_2 + \beta_D\xi\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} . \tag{6.35}$$

As in Section 6.2, we thus calculate natural elastic compliance and obtain the following due to flexure

$$f_{NEB} = \mathcal{L}_{MB}^{t} f^{\circ}_{NEB} \mathcal{L}_{MB} \tag{6.36}$$

and due to thrust

$$f_{NES} = \mathcal{L}_{MB}^{t} f^{\circ}_{NES} \mathcal{L}_{MB} , \tag{6.37}$$

where

$$f^{\circ}_{NEB} = \frac{1}{2}\int_{-1}^{+1} m_N^{t}\mathcal{L}_D^{t}\phi\mathcal{L}_D m_N \, d\xi \tag{6.38}$$

and

$$f^{\circ}_{NES} = \frac{1}{2}\int_{-1}^{+1} \gamma_N^{t}\mathcal{L}_D^{t}\psi\mathcal{L}_D\gamma_N \, d\xi . \tag{6.39}$$

The results from the extremely simple integrals are shown in Table 6.6. It is apparent that natural elastic compliance due to torsion is

$$f_{NET} = \frac{1}{GJ} \tag{6.40}$$ /168

where GJ is effective torsional rigidity.

We thus find total natural compliance to be

$$f_{NE} = \left[f_{NEB} + f_{NES} \quad f_{NET} \right] \tag{6.41}$$

We will of course only calculate elastic rigidity on the computer with the congruent transformation

$$k_E = a_N^t f_{NE}^{-1} a_N$$

since it would be considerably more difficult to program k_E directly.

For calculating geometric rigidity and the KEM, we recommend determining displacements within the element by application of the elastic rigidity matrix which has now been determined and integrating numerically. Even if the distribution of mass and normal force is not homogeneous, we need only a small amount of computing time, and thus avoid the necessity of setting up relatively complex matrices.

7. Calculation of Natural Oscillations of Rotor Blades by the Matrix Force Method /169

7.1. Introduction

The matrix force method[10] was developed while the matrix theory of statics was still in its initial stages. Due to the rapid development of the MDM, however, the MFM has been pushed into second place in the meantime. As a comparison of the above material with the following shows, the formulations of the MDM are considerably better suited to the use of large digital computers. If the standard routines for treating Boolean matrices are absent, however, it is possible to apply the MFM, which is closer to the conventional concepts of the engineer than the MDM, with approximately the same programming outlay, particularly for bar-shaped wing assemblies.

An additional disadvantage of the MFM is found in the solution of dynamic problems, however. The exceptional successes of the MDM are achieved primarily through application of the KEM, for whose calculation we require the displacements for the element such as are given from the outset in the MDM. Since a stress mode in the element is primarily employed in the MFM, it is generally not possible to consistently derive a mass matrix. In the beam problems at hand, however, idealization is effectively the same in

[10] Abbreviated MFM.

the force and displacement methods, since the EFT is applied in both cases. Thus the same results must be obtained within the precision of computation.

Since the MFM is dual to the MDM, we can be very brief in describing the basic principles of the MFM. The dual relationships can easily be derived by making use of the following "dictionary":

MDM	MFM
Displacement (strain)	Force (stress)
Force (stress)	Displacement (strain)
Rigidity	Compliance

Only the most important relationships will be repeated for the MFM; with each equation number, we also give that of the dual formulation for the MDM.

The unit load law[11] follows from the principle of virtual forces /170

$$R_1^t r = \int \sigma_1^t \varepsilon \, dV, \qquad (7.1, \text{D } 2.6)$$

where ε = given strain mode of system
r = sought displacements due to ε
R_1 = "unit load vector," i.e. a vector with unit loads in the direction of sought displacement r
σ_1 = stress mode statically compatible with R_1.

In order to set up compliance matrix f[12] for the element, the relation between element load P and the resultant stresses σ is converted into matrix form

$$\sigma = bP \qquad (7.1, \text{D } 2.9)$$

from which it follows that

$$f = \int b^t \mathcal{E}^{-1} b \, dV. \qquad (7.3, \text{D } 2.12)$$

The displacements ρ of the element are thus obtained from

---

[11] Abbreviated ULL.

[12] The subscript E is omitted in the MFM.

$$q = fP. \qquad (7.4,\ D\ 2.17)$$

The relationship between the load R on the wing assembly and the load P on the elements is obtained from equilibrium considerations and is written as

$$P = bR . \qquad (7.5,\ D\ 2.14)$$

We need not go into further details here [1, 2], since we are not considering statically indeterminate systems in the present work, and all degrees of freedom of the idealized structure are taken into consideration in column matrices r and R. Application of the ULL to the entire wing assembly yields

$$r = b^t q , \qquad (7.6,\ D\ 2.15)$$ /171

so , from equations (7.4), (7.5) and (7.6), we have

$$r = b^t q = b^t f b R = FR \qquad (7.7,\ D\ 2.19)$$

with the total compliance of the system

$$F = b^t f b , \qquad (7.8,\ D\ 2.20)$$

where f is of course the diagonal supermatrix of element compliances f_i here.

As in the case of the corresponding relationship in the MDM, it is desirable to break down matrix b for the loading of elements due to unit loads in terms of individual elements with

$$b = \{ b_1 \mid b_2 \cdots b_s \} \qquad (7.9)$$

and to calculate total compliance as the sum

$$F = \sum_i b_i f_i b_i \qquad (7.10,\ D\ 2.21)$$

since so much less storage space is required in the computer.

Here, too, a beam element with uniaxial flexure will serve as a simple example, without consideration given to shear deformation. Fig. 7.1 shows how the load and displacement vectors

$$P = \{M_1 \quad M_2\} \quad \text{and} \quad \varphi = \{\varphi_1 \quad \varphi_2\} \tag{7.11}$$

are selected for this element.

On the basis of the EFT, it is necessary to consider only the component σ_{zz} in stress vector σ, and we easily obtain

$$\sigma = \frac{y}{I}\left[\left(1 - \frac{z}{l}\right) \quad \frac{z}{l}\right] P , \tag{7.12, D 2.23}$$

i.e.

$$\sigma = \frac{y}{I}\left[\left(1 - \frac{z}{l}\right) \quad \frac{z}{l}\right] , \tag{7.12, D 2.24}$$

where, as in Section 2.5, /172

y = distance from neutral filament
I = moment of inertia of cross-sectional area.

The integration of equation (7.3) thus yields the known formula for the compliance of the beam element,

$$f = \frac{l}{6EI}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{7.13, D 2.25}$$

In order to allow an easy comparison with the displacement method, the generally valid formulation was also employed for this simple example, making use of the stress mode due to unit loads. In effect, of course, we need only apply the distribution of moments due to unit loads in beam problems

$$m = \left[\left(1 - \frac{z}{l}\right) \quad \frac{z}{l}\right] \tag{7.14}$$

from which we obtain

$$f = \int_0^l m^t \frac{1}{EI} m \, dz \tag{7.15}$$

In contrast to relationship (7.3), we have already integrated over the cross section of the beam in this equation.

The total compliance of the simple cantilever beam in Fig. 7.2 will be used to illustrate the application of matrix **b**, but we shall limit ourselves to the mutually corresponding loads and displacements which have been drawn in:

$$R' = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \qquad r = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}. \tag{7.16}$$

Written out in complete form, the matrix of element rigidities is

$$f = \left[\begin{array}{cc|cc} \frac{l_1}{3EI_1} & \frac{l_1}{6EI_1} & 0 & 0 \\ \frac{l_1}{6EI_1} & \frac{l_1}{3EI_1} & 0 & 0 \\ \hline 0 & 0 & \frac{l_2}{3EI_2} & \frac{l_2}{6EI_2} \\ 0 & 0 & \frac{l_2}{6EI_2} & \frac{l_2}{3EI_2} \end{array}\right]. \tag{7.17}$$

Matrix **b** must, according to equation (7.9) and the definition of **P**, Fig. 7.1, contain the section moments at the ends of each element due to loads R_1 and R_2. On the basis of elementary considerations, we find /173

$$b = \begin{array}{c} \begin{matrix} R_1 = 1 & R_2 = 1 \end{matrix} \\ \left[\begin{array}{cc} l_1 & l_1 + l_2 \\ 0 & l_2 \\ \hline 0 & l_2 \\ 0 & 0 \end{array}\right] \end{array}. \tag{7.18}$$

We obtain total compliance with the aid of the congruent transformation in equation (7.7) to be

$$F = b^{T} f b = \begin{bmatrix} \frac{l_1^3}{3EI_1} & \frac{2l_1^3 + 3l_1^2 l_2}{6EI_1} \\ \frac{2l_1^3 + 3l_1^2 l_2}{6EI_1} & \frac{l_1^3 + 3l_1^2 l_2 + 3l_1 l_2^2}{3EI_1} + \frac{l_2^3}{3EI_2} \end{bmatrix} \quad (7.19)$$

7.2. Calculation of Natural Oscillations

7.2.1. Calculation of Natural Oscillations Without Normal Forces

The case of an articulated connection, relatively difficult for treatment by the MFM, is the most important one in practice for the calculation of rotor blade natural oscillations. Such a rotor blade represents a statically underdetermined system. In the MDM, it does not matter whether a system is statically determined, undetermined or underdetermined, since the rigidity matrices k of the elements always yield an equilibrium system of forces P for any displacement ρ. In the MFM, however, there is no compliance matrix for a statically underdetermined system. As the following discussion indicates, this difficulty can easily be circumvented, however. The reader can easily derive the relationships himself for the simpler case of a rigid blade connection.

The basic matrices for calculating natural oscillations of a rotor blade with uniaxial flexure are explained in Fig. 7.3.

The compliance of the i^{th} element, according to equation (7.13), becomes /174

$$f_i = \frac{l_i}{6E_i I_i} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad (7.20)$$

All f_i are collected again in the diagonal supermatrix

$$f = \begin{bmatrix} f_1 & f_2 & \cdots & f_s \end{bmatrix} \quad (7.21)$$

Since a total compliance does not exist for the articulated rotor blade, as already mentioned, compliance will first be calculated for a restrained blade; we call this F'. The associated matrix b of section moments at the ends of the elements due to unit loads and moments can be taken from Table 7.1.

We use this matrix **b** to obtain the compliance of the restrained rotor blade through the application of equation (7.8). We can now formulate displacement **r** as the sum of rigid-body displacement due to rotation of the blade at the joint by an angle ψ and the elastic deformations experienced by the restrained blade

$$r = L\psi + F'R \quad , \qquad (7.22)$$

rigid elastic

where, as can easily be seen,

$$L = \{L_1 \quad L_2 \cdots L_s \underbrace{1 \; 1 \cdots 1}_{s+1}\}. \qquad (7.23)$$

In order to determine the new unknown ψ, we shall make use of the condition for moment equilibrium, referred to the joint, which can be written in matrix form as

$$L^t R = 0. \qquad (7.24)$$

The KEM, still required for calculating the natural oscillations, can be taken from the deformation method, since idealization is effectively the same. On the other hand, it is not possible to use the elegant buildup method with

$$M = a^t m a. \qquad (7.25)$$ /175

Table 7.2 shows how the mass matrix must be set up in agreement with the displacement vector **r** used here.

For harmonic oscillation of angular frequency ω we obtain inertial forces

$$R_T = \omega^2 M r \quad , \qquad (7.26)$$

which, substituted into equation (7.22), yield the relationship

$$r = L\psi + \omega^2 F' M r \qquad (7.27)$$

The condition for moment equilibrium (7.24) applied to the inertial forces (7.26) thus yields the following from (7.27) for the angle of rotation:

$$\psi = -\omega^2\left[L^tML\right]^{-1}L^tMF'Mr\,. \tag{7.28}$$

The angle of rotation ψ for rigid rotation, now known, is substituted into equation (7.27) and yields

$$r = \omega^2\left[I - L(L^tML)^{-1}L^tM\right]F'Mr \tag{7.29}$$

or, transformed,

$$\left\{I - \omega^2\left[I - L(L^tML)^{-1}L^tM\right]F'M\right\}r = 0\,, \tag{7.30}$$

where the dynamic matrix in the force method at hand reads

$$D = \left[I - L(L^tML)^{-1}L^tM\right]F'M \tag{7.30a}$$

We have thus again obtained an eigenvalue problem, which can be solved by the familiar method. Observe that the dynamic matrix in the MFM need not be identical to that in the MDM (equation (5.5)). In order to calculate the modes of natural oscillation r_i, we can also use equation (7.27), from which we obtain /176

$$r_i = \left(I - \omega_i^2F'M\right)^{-1}L\psi\,. \tag{7.31}$$

We obtain section moments P_i in each mode of natural oscillation from

$$P_i = \omega_i^2bMr_i\,. \tag{7.32}$$

7.2.2. Simplified Consideration of Normal Forces

The additional loading of elements due to the normal forces resulting from rotation of the blades is taken to be a linear

function of the displacements, as in the MDM. Equation (7.5) is thus expanded to

$$P = bR - cr. \tag{7.33}$$

The columns in matrix c thus contain the section moments at the ends of the individual elements as produced by unit displacements under given normal forces. Since we provisionally limit ourselves to the application of simple element compliance from (7.21), only a linear behavior on the part of moments within each element resulting from the normal forces can be taken into consideration, meaning that the displacement in each element can only be taken as a linear function for calculating these moments. Angles of rotation ϕ_i must therefore remain unconsidered for the time being, so

$$c = [c_r \quad 0] \quad . \qquad (2s \times 2s + 1) \tag{7.34}$$

Table 7.3 shows how matrix c_r must be built up. The centrifugal forces Z_i used there are determined from the mass m_i lying in the high-tension area of the i^{th} node using

$$Z_i = \Omega^2 m_i L_i \tag{7.35}$$

where Ω is the angular velocity of the rotor. The high-tension area must be determined in such a manner that the node is the center of gravity of the mass m_i concentrated in it. The section normal forces N_i are calculated from

$$N_i = \sum_{j=i}^{s} Z_j \tag{7.35a}$$

By applying the ULL and adding the rigid body displacements, /177 we now obtain the following in place of (7.22)

$$r = L\psi + b^t f b R - b^t f c r. \tag{7.36}$$

From this it follows that

$$(I + b^t f c)r = L\psi + F'R \tag{7.37}$$

or

$$r = T^{-1}(L\psi + F'R) \tag{7.38}$$

with the so-called geometric coupling matrix

$$T = I + b^t f c . \tag{7.39}$$

It should be mentioned in passing that the coupling loads can also be calculated from T using the condition

$$\det T = 0. \tag{7.40}$$

Determination of the natural oscillation of the articulated rotor blade is simplified, when we take normal forces into consideration, by the fact that a statically determinate system then exists. The unknown angle of rotation ψ can therefore be calculated directly from the condition for moment equilibrium in the joint, which is now written in matrix form as

$$L^t R + Z^t r = 0 \tag{7.41}$$

where

$$Z = \{Z_1\ Z_2 \cdots Z_s \mid \underbrace{0\ 0 \cdots 0}_{s+1}\} \tag{7.42}$$

From equations (7.38) and (7.41), it follows that

$$\psi = (Z^t T^{-1} L)^{-1}(L^t - Z^t T^{-1} F')R . \tag{7.43}$$

If this relationship is substituted into (7.38), we obtain /178

$$r = T^{-1}\left[L(Z^t T^{-1} L)^{-1}(L^t - Z^t T^{-1} F') + F'\right]R . \tag{7.44}$$

The compliance of the rotatably mounted rotor blade with normal forces is thus

$$F = T^{-1}\left[L\,(Z'T^{-1}L)^{-1}(L' - Z'T^{-1}F') + F'\right] . \qquad (7.45)$$

Matrix **F** is of course symmetrical, as required by Maxwell's reciprocity theorem; this can easily be checked.

Since the compliance matrix **F** is now known, we easily arrive at the conventional form of the eigenvalue problem in the MFM [1]:

$$(I - \omega^2 FM)r = 0 . \qquad (7.46)$$

Nothing else need be said regarding solution of the eigenvalue problem. We obtain the section bending moments of the i^{th} mode of natural oscillation from

$$P_i = bR_r - cr_i = (\omega^2 bM - c)r_i . \qquad (7.47)$$

7.2.3. Consideration of Normal Forces in the Assumed Elastic Deformation Modes of the Elements

In this section we show how the effect of the normal force is taken into consideration in the MFM if we use an idealization which better matches reality for the deformation modes of the elements. The deformation modes show in Fig. 2.2 will serve as an example. Since these elastic curves are third-order parabolas, the behavior of moment must also be given by curves of third order, meaning that combinations of four linearly independent moment distributions are permissible, as shown in Table 7.5. The first two forms are determined by the section moments at the ends of the element. These moments at the ends of the elements are -- as we can easily conceive -- not functions of the shape of the elastic curve, so we can use the same matrix c_r (Table 7.3) as in the preceding section in order to calculate them. The moments at the ends are zero for the last two distributions, so it is desirable here to directly employ the normal force operant in the deformed elements as a given quantity, which would also have been possible with the first forms, of course. The normal forces operant in deformed elements i and i + 1 due to rotations $\phi_i = 1$ are given by matrix c_ϕ from Table 7.4. The displacements ρ_i of the element induced by the above four moment

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distributions are obtained with the aid of expanded compliance matrix f_e derived in Table7.5. If we now use

$$f_{e1i} = \frac{l_i}{60 E_i I_i}\begin{bmatrix} 21 & 9 \\ 9 & 21 \end{bmatrix} \quad \text{and} \quad f_{e2i} = \frac{l_i^2}{60 E_i I_i}\begin{bmatrix} -3 & 2 \\ -2 & 3 \end{bmatrix} \tag{7.48}$$

to designate the partial compliances of the i^{th} element, the expanded compliance matrix for all elements of the rotor blade becomes

$$f_e = [f_{e1} \quad f_{e2}], \tag{7.49}$$

where

$$f_{e1} = \lceil f_{e1i} \rfloor \quad \text{and} \quad f_{e2} = \lceil f_{e2i} \rfloor . \tag{7.50}$$

Expanded matrix c_e now reads

$$c_e = \lceil c_r \quad c_\varphi \rfloor \quad (4s \times 2s + 1) \tag{7.51}$$

in place of equation (7.34), and thus equation (7.37) now becomes

$$(I + b^t f_e c_e) r = L\psi + F^t R , \tag{7.52}$$

so the geometric coupling matrix is now

$$T = (I + b^t f_e c_e) \tag{7.53}$$

The further calculation of natural oscillations proceeds as in the preceding section.

7.3. Examples

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The application of geometric coupling matrix T will now be demonstrated in the purely static example of a restrained flexible beam under a normal force (see Fig. 7.4) with one element.

As we know from the example in Section 7.1, the following applies to this cantilever beam without normal force:

$$f = \frac{l}{6EI}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \tag{7.54)(7.55}$$

from which

$$F' = \frac{l^3}{6EI}\begin{bmatrix} 2 & 3/l \\ 3/l & 6/l^2 \end{bmatrix} \tag{7.56}$$

The expanded compliance matrix for this element is

$$f_e = \frac{1}{60EI}\begin{bmatrix} 21 & 9 & -3l & 2l \\ 9 & 21 & -2l & 3l \end{bmatrix}. \tag{7.57}$$

We easily find matrix c_e to be

$$c_e = N\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{7.58}$$

from which we find the geometric coupling matrix to be

$$T = I + b^t f_e c_e = \begin{bmatrix} 1+21K & 2Kl \\ 30\frac{K}{l} & 1+5K \end{bmatrix}, \tag{7.59}$$

with the dimensionless quantity

$$K = \frac{Nl^2}{60EI}. \tag{7.60}$$

For the restrained beam with $\psi = 0$, we obtain the following directly from equation (7.38):

$$F = T^{-1}F' = \frac{l^3}{6EI(1+26K+45K^2)}\begin{bmatrix} 2(1+2K) & \frac{3}{l}(1+K) \\ \frac{3}{l}(1+K) & \frac{6}{l^2}(1+6K) \end{bmatrix} \tag{7.61}$$

For a force R = 1, we thus obtain the following displacement by the MFM: /181

$$r_{MFM} = \frac{R l^3}{3EI} \frac{1 + 2\varkappa}{1 + 26\varkappa + 45\varkappa^2} . \tag{7.62}$$

Solution of this problem with the aid of the corresponding differential equation from EFT yields

$$r_{EFT} = \frac{R l}{N} \left(1 - \sqrt{\frac{EI}{Nl^2}} \tanh \sqrt{\frac{Nl^2}{EI}}\right) . \tag{7.63}$$

Fig. 7.5 shows relative error

$$D = (r_{EFT} - r_{MFM})/r_{EFT} \tag{7.64}$$

as a function of the dimensionless quantity $Nl^2/(EI)$ in percent. We see that even for relatively large normal forces, the simple idealization of the cantilever beam with one element still gives quite good results.

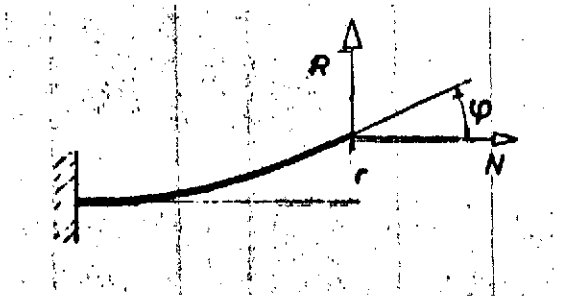

Fig. 7.4. Example of a restrained beam with one element under a transverse and a normal force.

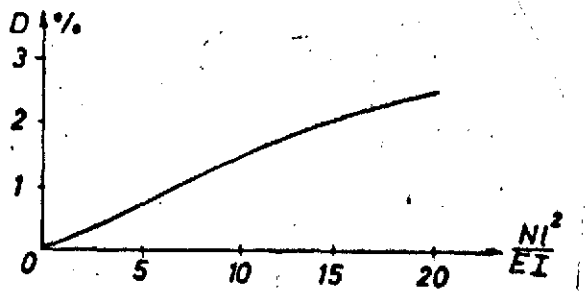

Fig. 7.5. Error curve for flexure in the example in Fig. 7.4.

Table 7.6 shows a simplified example from among the many rotor blades which we have already calculated by the MFM.

The analytical solution has also been determined for natural oscillations of this homogeneous flexible beam. In addition, experimentally determined natural frequencies without normal force are listed in the Table.

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Please Note:

For clarity, dots are used in place of zeros in matrices in the figures and tables.

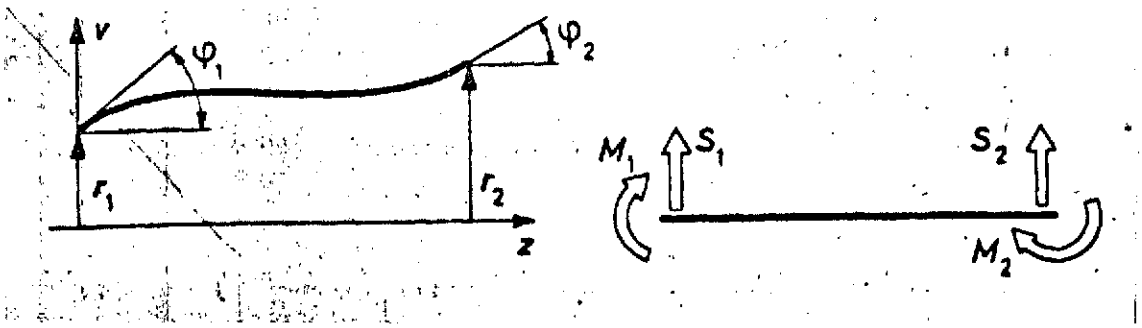


Fig. 2.1. Displacements and forces for a homogeneous beam element with uniaxial flexure.

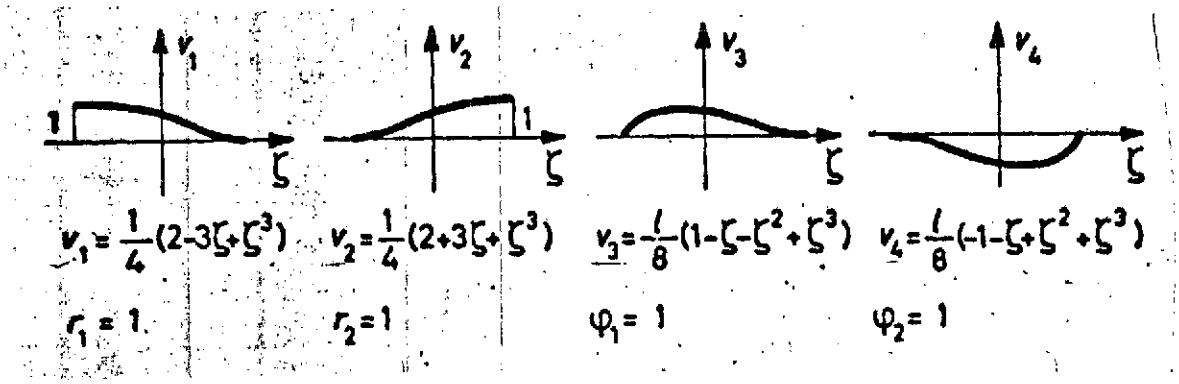


Fig. 2.2. Elastic curves for unit displacement modes of homogeneous beam element.

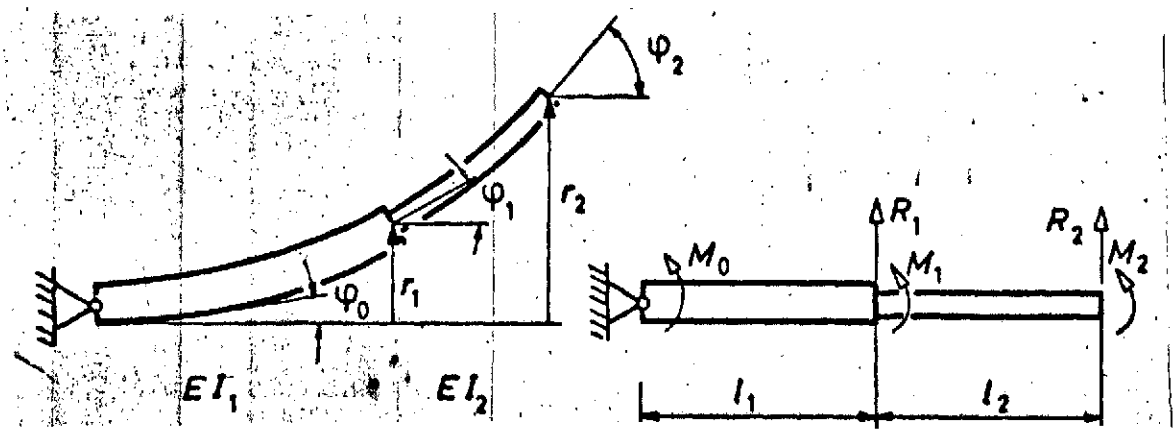


Fig. 2.3. Example of calculation of total rigidity.

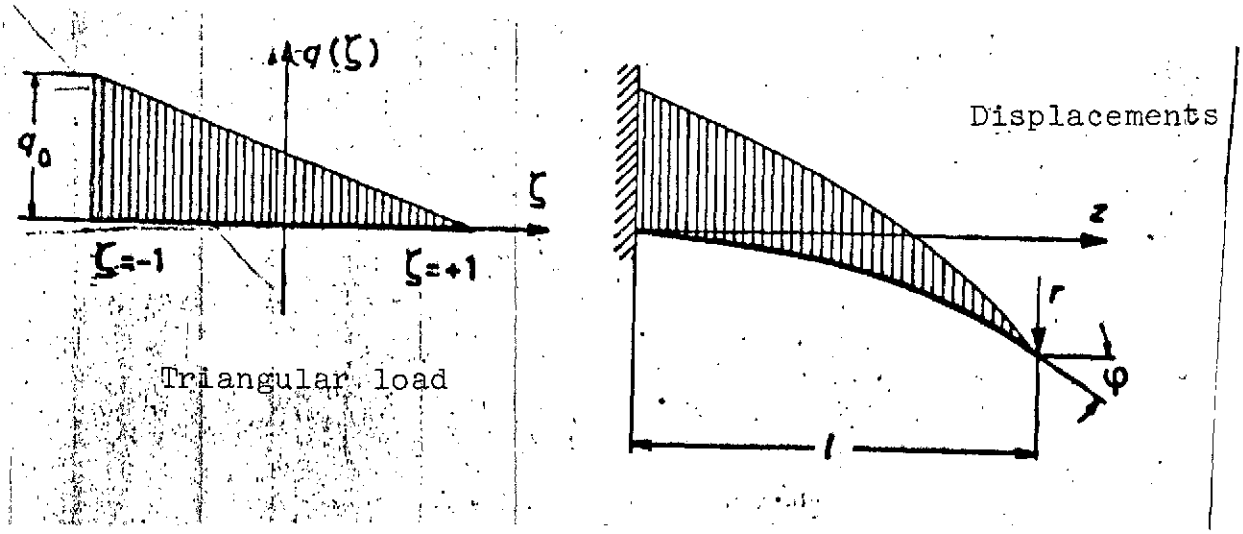


Fig. 3.1. Beam element with triangular load

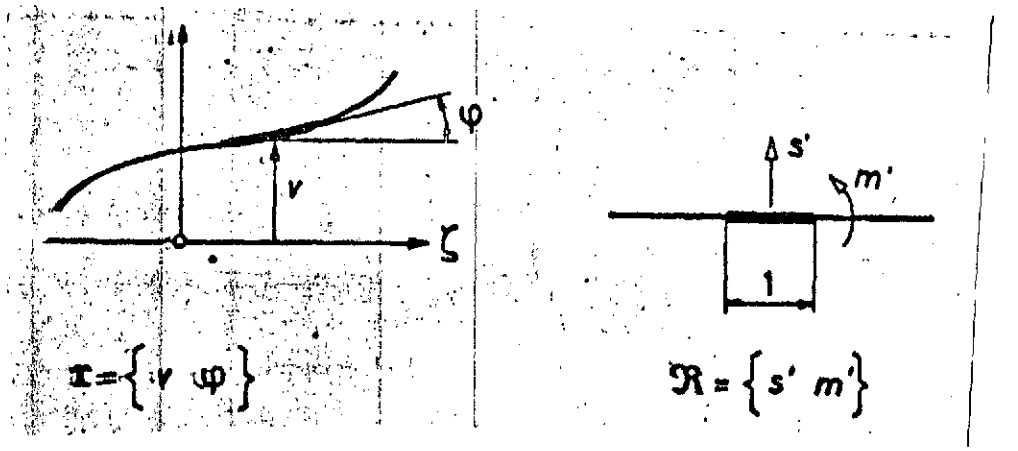


Fig. 3.2. Displacements and forces for a longitudinal element.

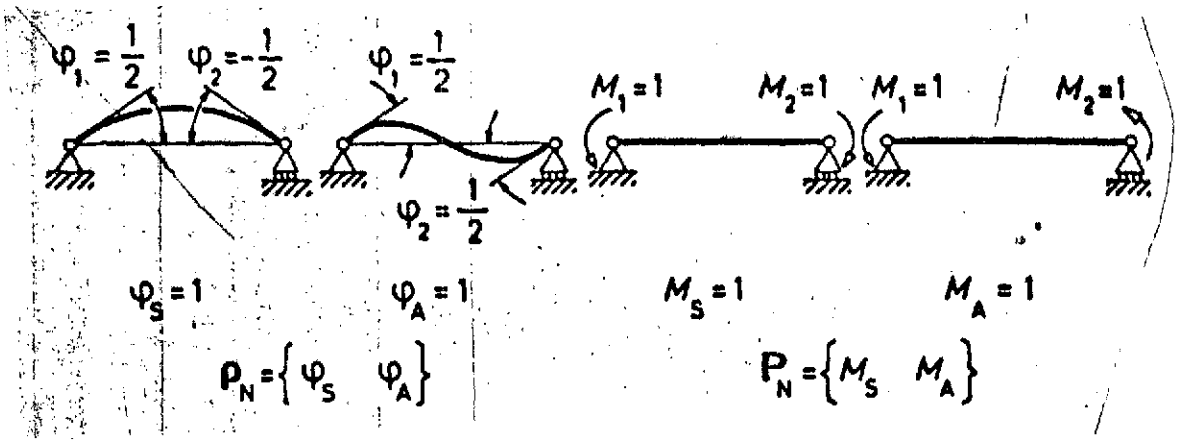


Fig. 4.1. Natural displacements and forces for a beam element.

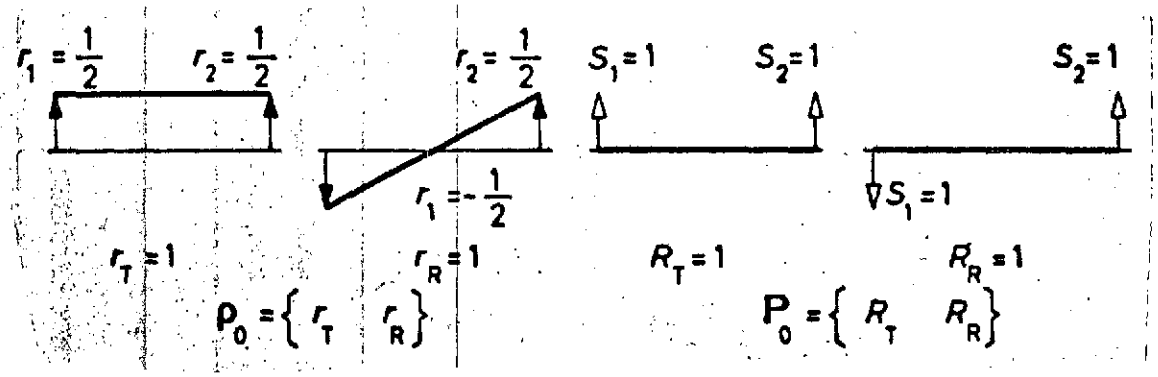


Fig. 4.2. Rigid body displacements and resultant forces for a beam element.

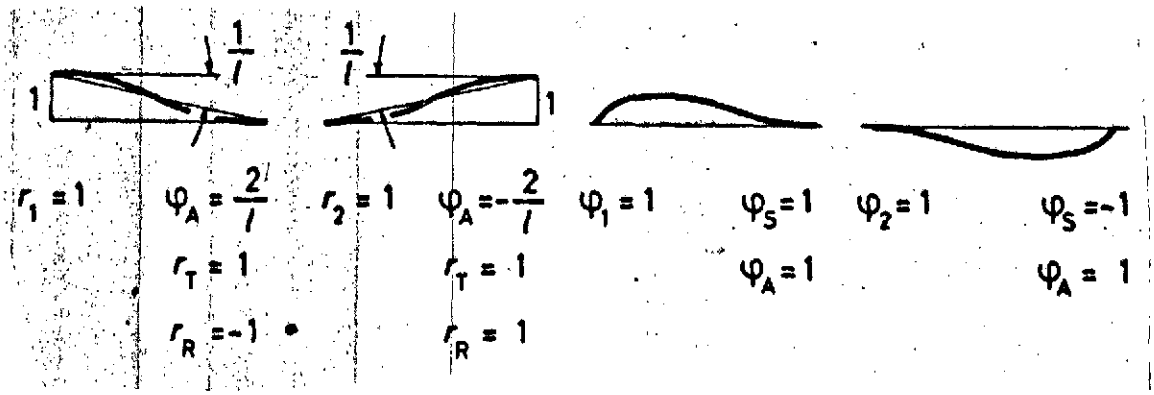


Fig. 4.3. Breakdown of assumed deformation modes of beam element into natural and rigid body displacements.

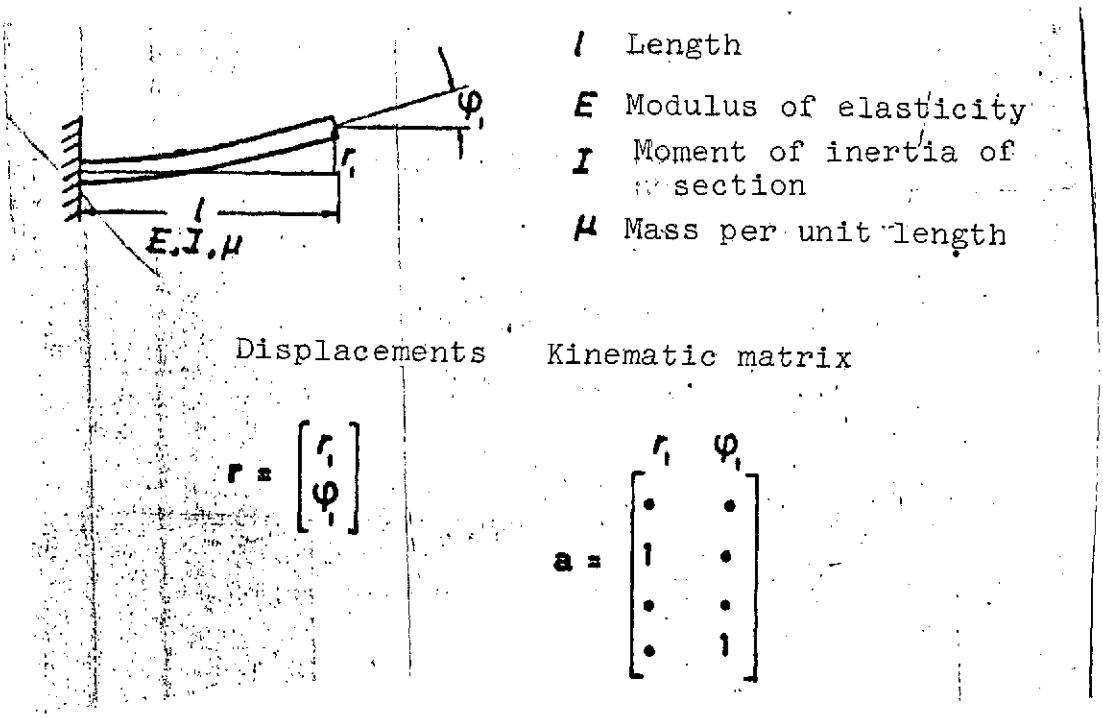


Fig. 5.1. Restrained beam -- idealization with one element.

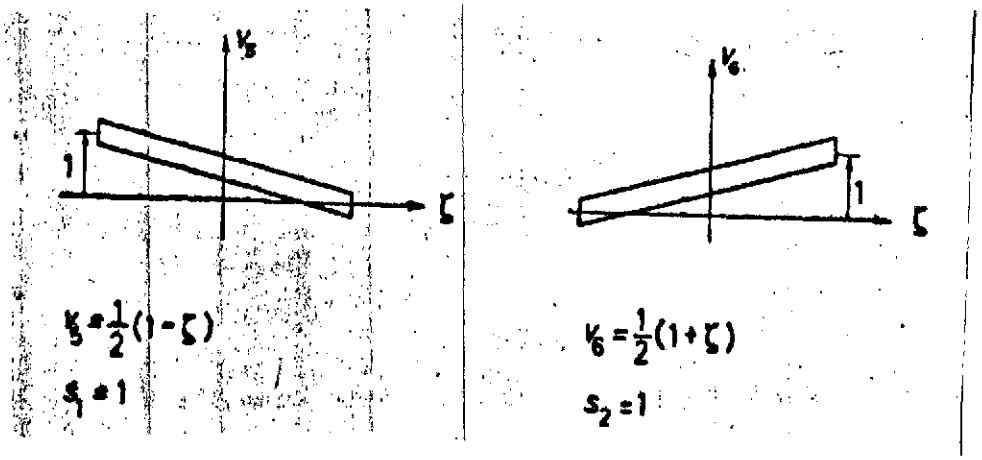


Fig. 6.1. Displacement modes of beam element due to pure shear deformation. See Fig. 2.2 for v_1 through v_4.

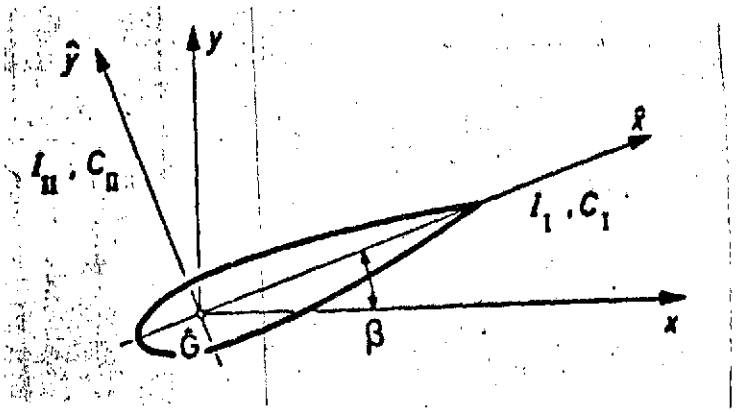


Fig. 6.2. Principal axes of a twisted rotor blade element.

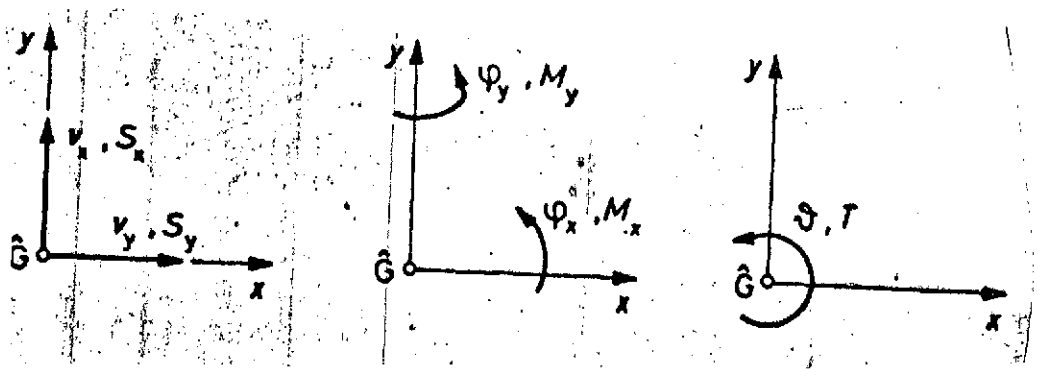


Fig. 6.3. Displacements and forces under biaxial flexure and torsion.

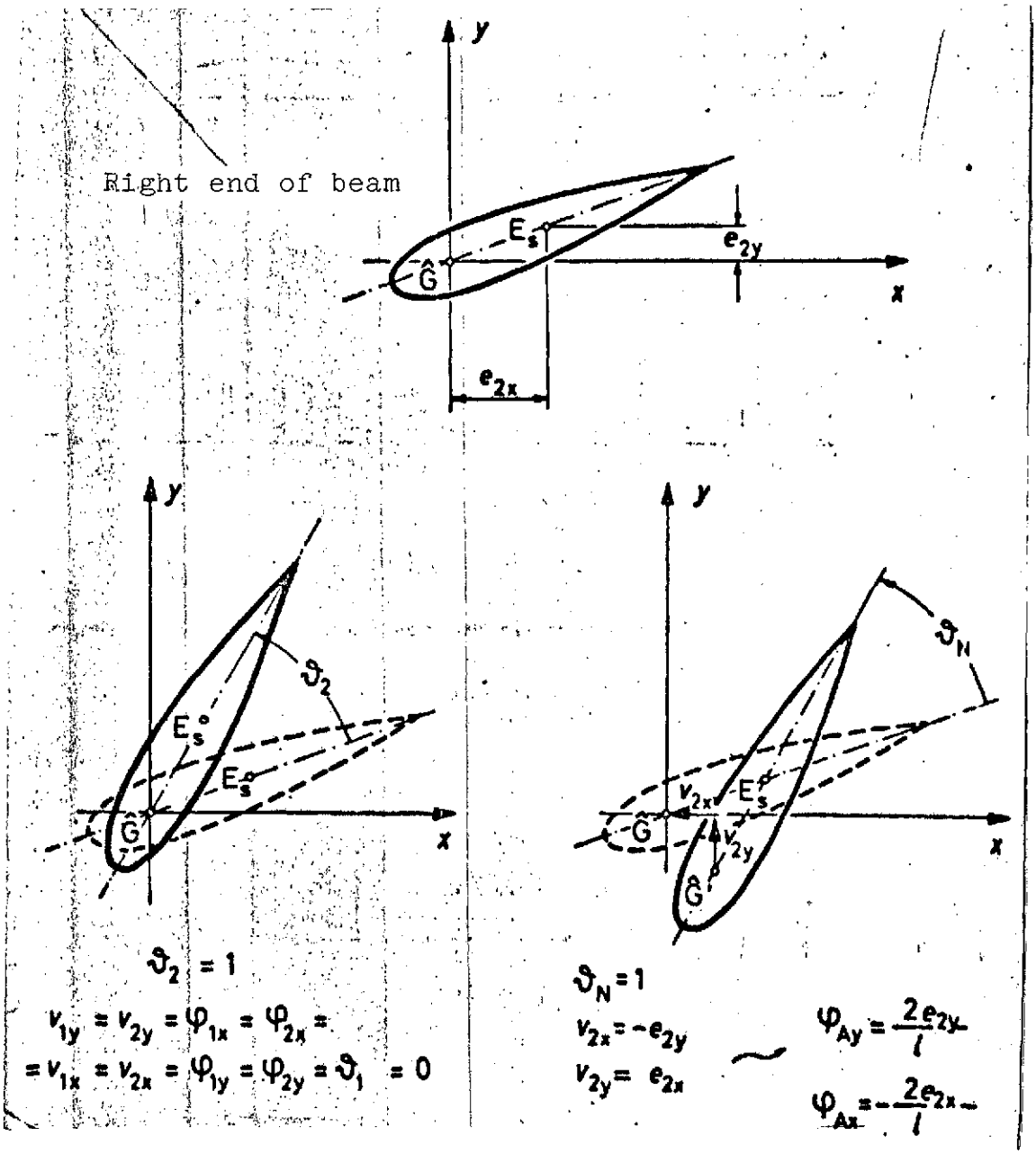


Fig. 6.4. Natural deformation mode due to unit displacement.

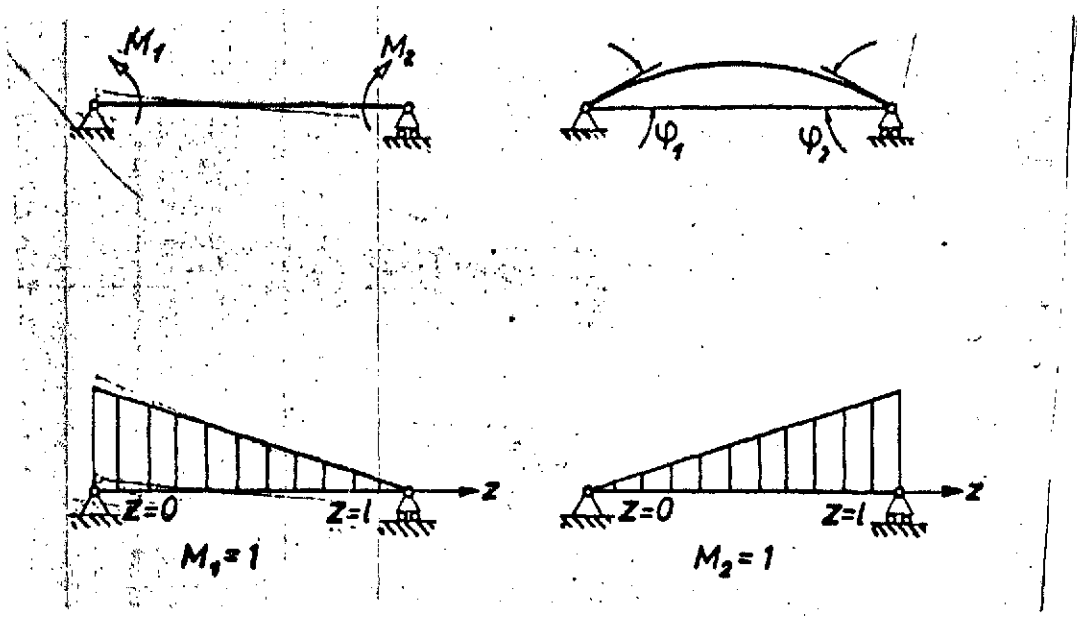


Fig. 7.1. Loads and displacements of an element and the associated moment distributions.

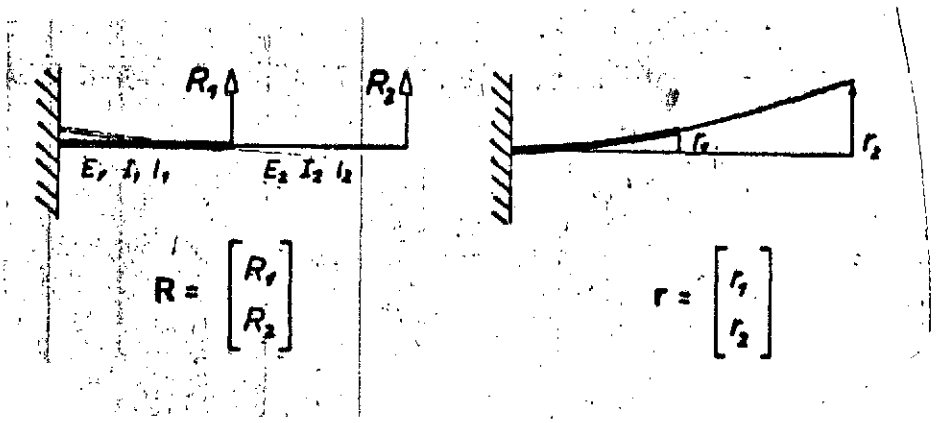


Fig. 7.2. Example of a cantilever beam with two elements.

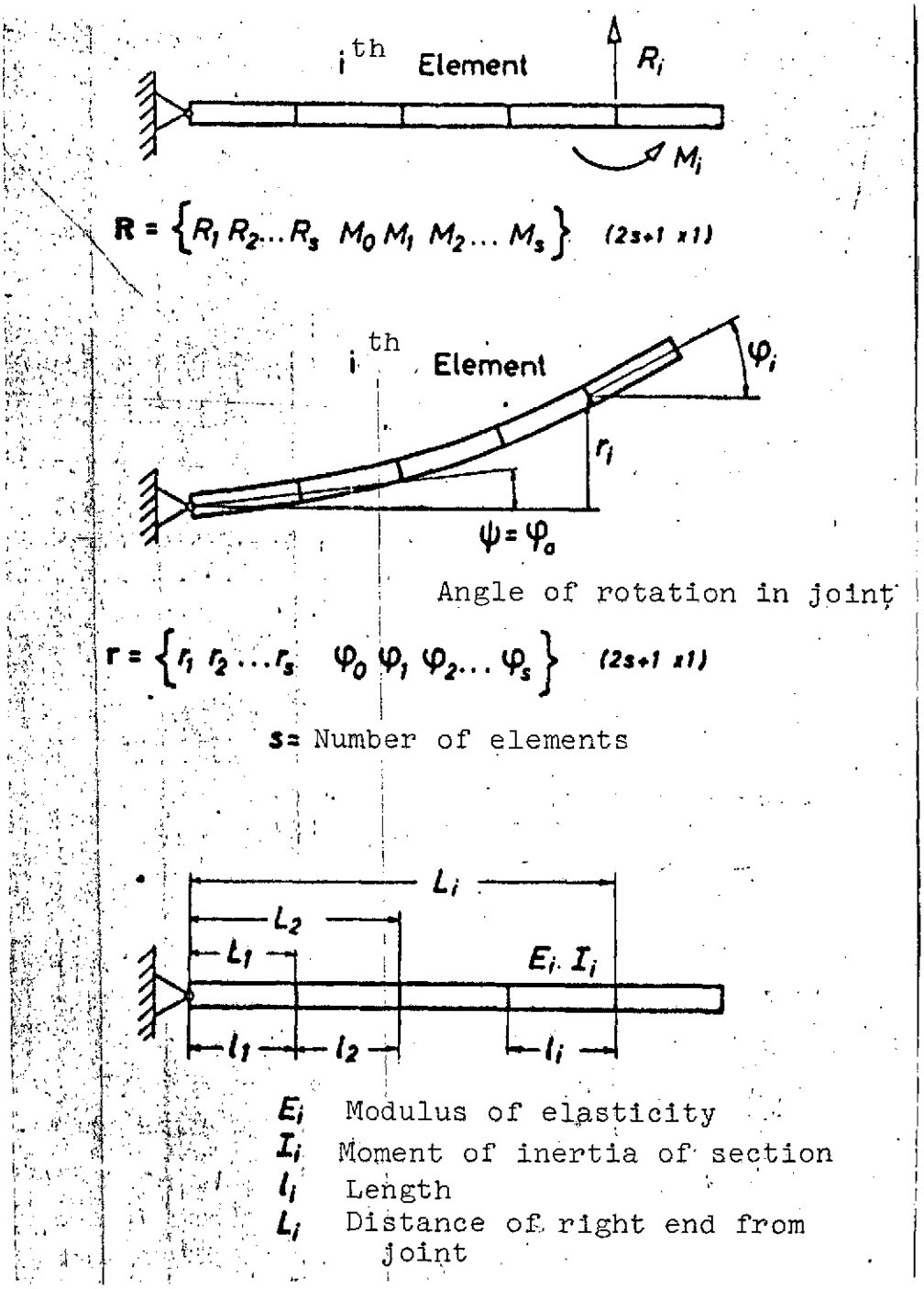


Fig. 7.3. Assignment of force and displacement vectors and characteristic values for individual elements.

TABLE 2.1. KINEMATIC MATRIX a AND TOTAL RIGIDITY K_E FOR THE EXAMPLE IN FIG. 2.3 /194

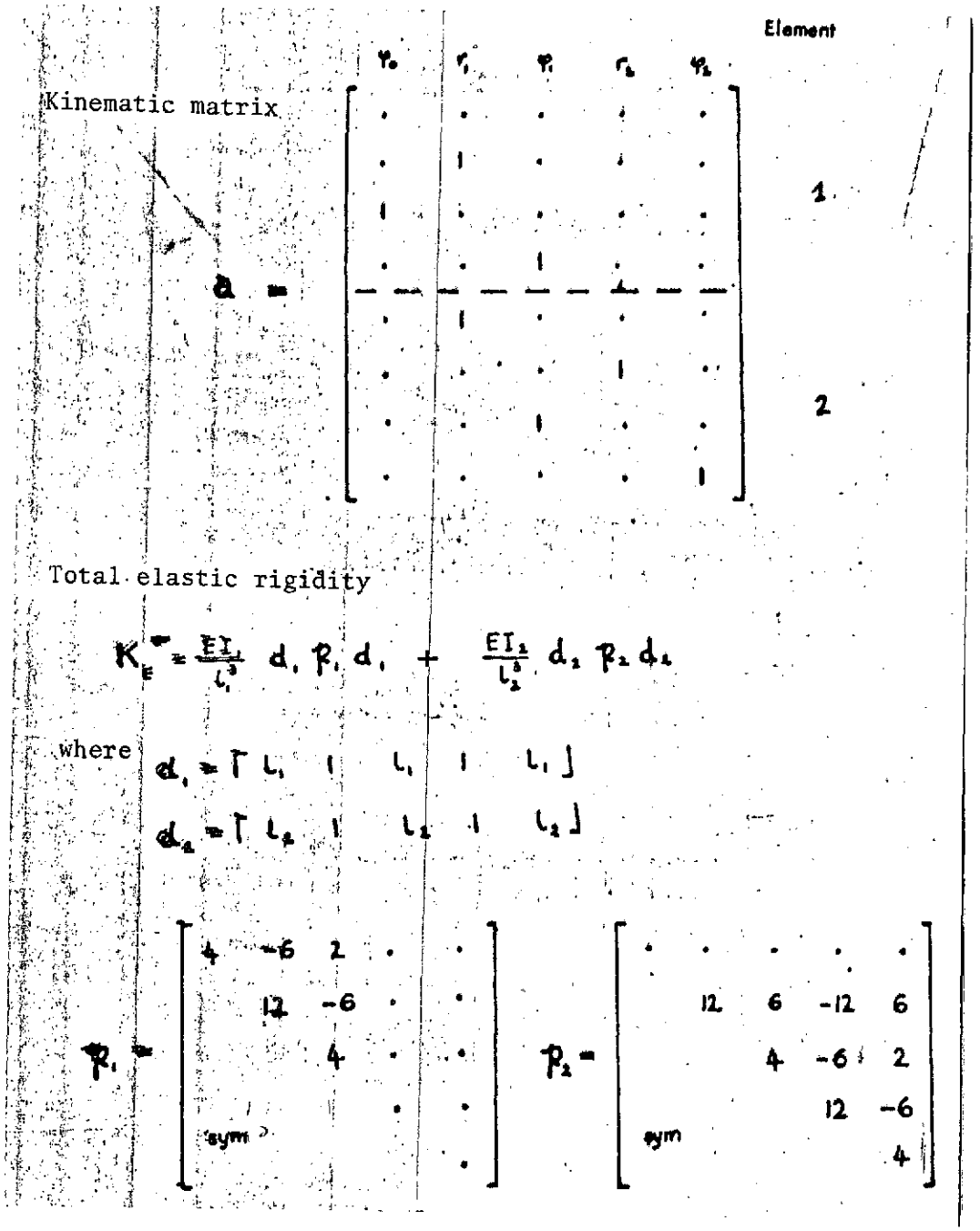
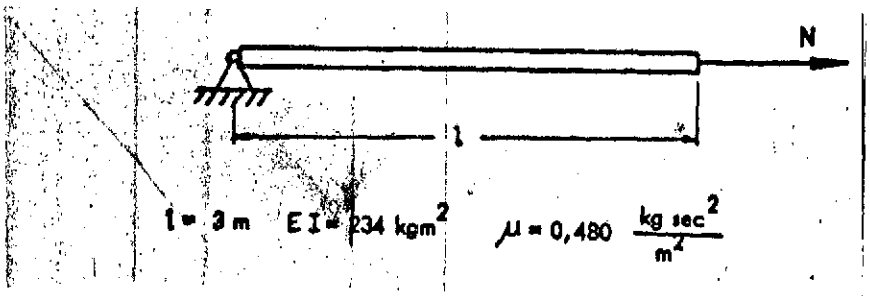

Element
Kinematic matrix
a =
1
2
Total elastic rigidity
$K_E = \frac{EI_1}{l_1^3} d_1 P_1 d_1 + \frac{EI_2}{l_2^3} d_2 P_2 d_2$
where
$d_1 = [\, l_1 \quad 1 \quad l_1 \quad 1 \quad l_1 \,]$
$d_2 = [\, l_2 \quad 1 \quad l_2 \quad 1 \quad l_2 \,]$
$P_1 =$ 4 -6 2 12 -6 4 sym
$P_2 =$ 12 6 -12 6 4 -6 2 12 -6 4 sym

TABLE 5.1. NATURAL FREQUENCIES OF A HOMOGENEOUS FLEXIBLE BEAM UNDER CONSTANT NORMAL FORCE; COMPARISON WITH ANALYTICAL SOLUTION



Comparison of results:

First line 5 elements
Second line 10 element
Third line differential equation

ν_n (Hz) \ N (kp)	0	260	520	780	1040
ν_1	-	2.038	2.837	3.446	3.960
	-	2.040	2.838	3.447	3.960
	-	2.036	2.832	3.440	3.949
ν_2	6.021	8.748	10.715	12.324	13.716
	6.020	8.746	10.712	12.320	13.712
	6.019	8.745	10.711	12.319	13.711
ν_3	19.553	22.152	24.448	26.518	28.413
	19.510	22.104	24.396	26.462	28.354
	19.506	22.101	24.392	26.458	28.349
ν_4	41.074	43.526	45.840	48.034	50.121
	40.727	43.177	45.486	47.673	49.753
	40.698	43.147	45.456	47.642	49.722
ν_5	71.928	74.320	76.639	78.893	81.079
	69.737	72.087	74.360	76.561	78.697
	69.597	71.946	74.219	76.420	78.555

TABLE 6.1. NUMERICAL MATRICES FOR INHOMOGENEOUS ROTOR BLADE ELEMENT

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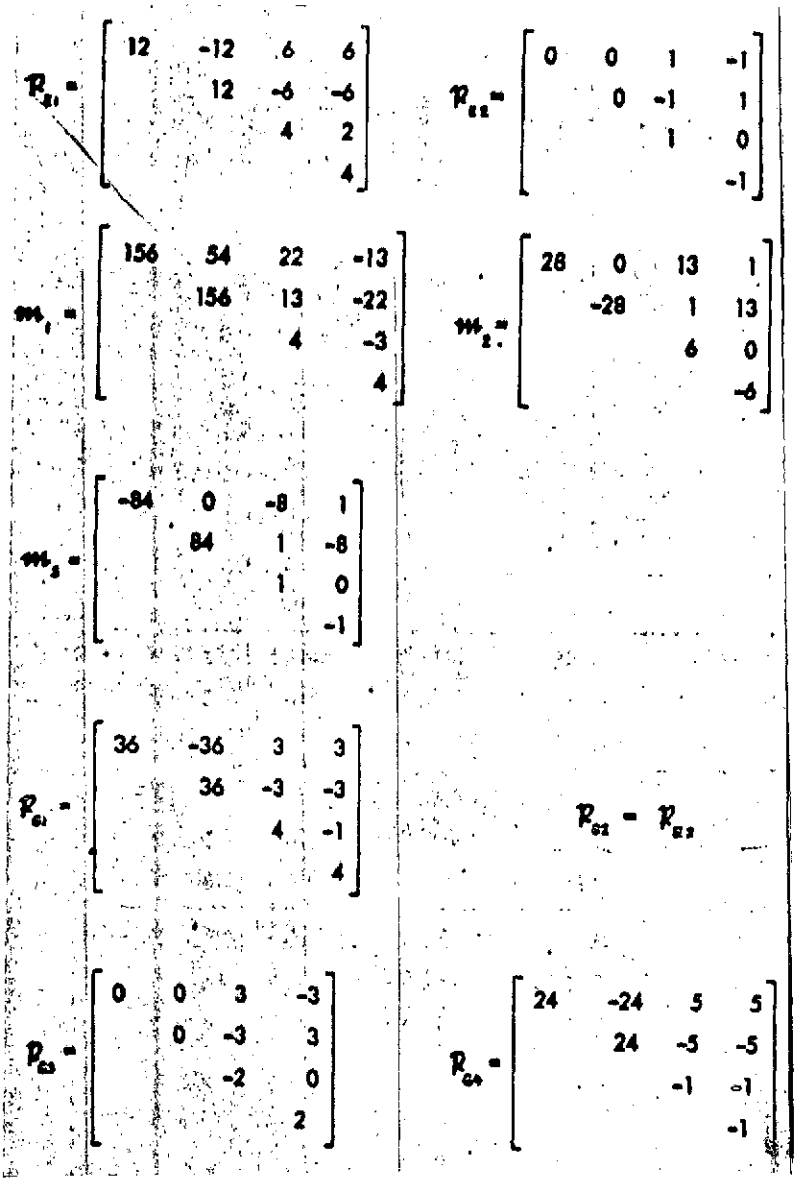
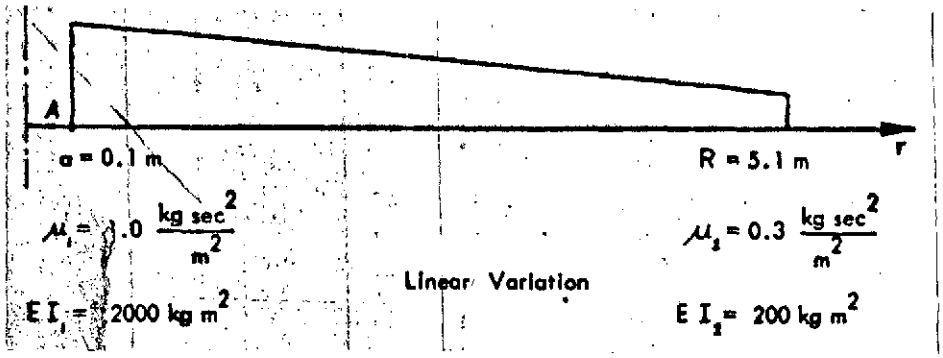

$P_{11} = \begin{bmatrix} 12 & -12 & 6 & 6 \\ & 12 & -6 & -6 \\ & & 4 & 2 \\ & & & 4 \end{bmatrix}$
$P_{12} = \begin{bmatrix} 0 & 0 & 1 & -1 \\ & 0 & -1 & 1 \\ & & 1 & 0 \\ & & & -1 \end{bmatrix}$
$M_1 = \begin{bmatrix} 156 & 54 & 22 & -13 \\ & 156 & 13 & -22 \\ & & 4 & -3 \\ & & & 4 \end{bmatrix}$
$M_2 = \begin{bmatrix} 28 & 0 & 13 & 1 \\ & -28 & 1 & 13 \\ & & 6 & 0 \\ & & & -6 \end{bmatrix}$
$M_3 = \begin{bmatrix} -84 & 0 & -8 & 1 \\ & 84 & 1 & -8 \\ & & 1 & 0 \\ & & & -1 \end{bmatrix}$
$P_{21} = \begin{bmatrix} 36 & -36 & 3 & 3 \\ & 36 & -3 & -3 \\ & & 4 & -1 \\ & & & 4 \end{bmatrix}$
$P_{22} = P_{12}$
$P_{23} = \begin{bmatrix} 0 & 0 & 3 & -3 \\ & 0 & -3 & 3 \\ & & -2 & 0 \\ & & & 2 \end{bmatrix}$
$P_{24} = \begin{bmatrix} 24 & -24 & 5 & 5 \\ & 24 & -5 & -5 \\ & & -1 & -1 \\ & & & -1 \end{bmatrix}$

All matrices are symmetrical

TABLE 6.2. NATURAL FREQUENCIES OF A TAPERED ROTOR BLADE; CONVERGENCE AS SUBDIVISION BECOMES FINER /197

Tapered rotor blade

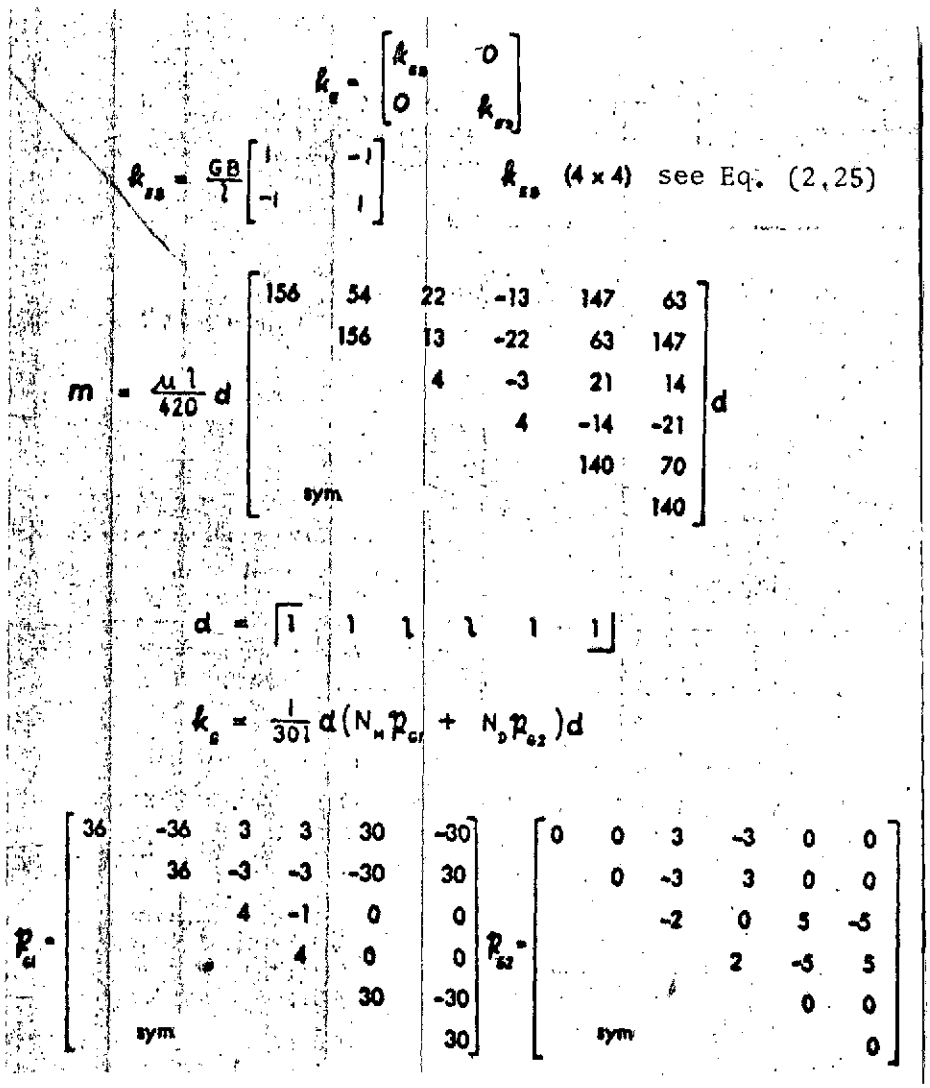


First line 5 elements
Second line 10 elements
Third line 15 elements

u = velocity of blade tip due to rotation

f_n (Hz) \ u ($\frac{m}{sec}$)	0	60	120	180	240
	4.452	6.126	9.511	13.357	17.360
ν_1	4.489	6.154	9.528	13.365	17.359
	4.491	6.159	9.531	13.366	17.358
	13.088	14.683	18.645	23.786	29.476
ν_2	13.266	14.853	18.804	23.937	29.619
	13.295	14.880	18.827	23.955	29.632
	26.429	27.973	32.144	38.051	44.981
ν_3	26.844	28.397	32.591	38.526	45.481
	26.909	28.459	32.651	38.585	45.538
	44.715	46.246	50.557	56.999	64.904
ν_4	45.297	46.829	51.137	57.561	65.432
	45.384	46.919	51.230	57.659	65.538
	73.824	75.254	79.379	85.781	93.959
ν_5	68.717	70.236	74.596	81.311	89.812
	68.760	70.285	74.660	81.397	89.925

TABLE 6.3. 6 x 6 RIGIDITY AND MASS MATRICES FOR HOMOGENEOUS BEAM ELEMENT WITH SHEAR DEFORMATION /198



$$k_e = \begin{bmatrix} k_{ss} & 0 \\ 0 & k_{rr} \end{bmatrix}$$

$$k_{ss} = \frac{GB}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad k_{ss}\ (4 \times 4)\ \text{see Eq. (2.25)}$$

$$m = \frac{\rho A l}{420} d \begin{bmatrix} 156 & 54 & 22 & -13 & 147 & 63 \\ & 156 & 13 & -22 & 63 & 147 \\ & & 4 & -3 & 21 & 14 \\ & & & 4 & -14 & -21 \\ & & & & 140 & 70 \\ & \text{sym} & & & & 140 \end{bmatrix} d$$

$$d = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$k_g = \frac{1}{30l} d\,(N_M P_{G1} + N_D P_{G2})\,d$$

$$P_{G1} = \begin{bmatrix} 36 & -36 & 3 & 3 & 30 & -30 \\ & 36 & -3 & -3 & -30 & 30 \\ & & 4 & -1 & 0 & 0 \\ & & & 4 & 0 & 0 \\ & & & & 30 & -30 \\ & \text{sym} & & & & 30 \end{bmatrix} \qquad P_{G2} = \begin{bmatrix} 0 & 0 & 3 & -3 & 0 & 0 \\ & 0 & -3 & 3 & 0 & 0 \\ & & -2 & 0 & 5 & -5 \\ & & & 2 & -5 & 5 \\ & & & & 0 & 0 \\ & \text{sym} & & & & 0 \end{bmatrix}$$

TABLE 6.4. NATURAL FREQUENCIES OF A BEAM, TAKING SHEAR DEFORMATION AND ROTATIONAL INERTIA INTO CONSIDERATION

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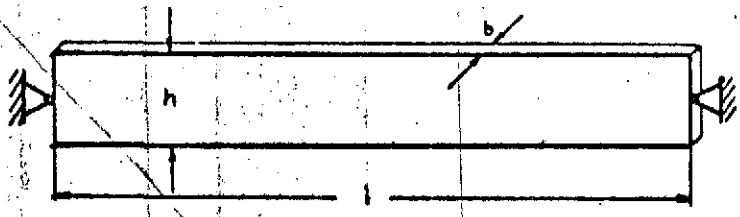

$l = 40$ in $h = 6$ in $b = 2$ in $F = b \cdot h = 12$ in^2

Modulus of elasticity $E = 30 \cdot 10^6$ lbf/in^2 shear mod. $G = 11.5 \cdot 10^6$ lbf/in^2

Density $\gamma = 0.28$ lbf/in^3

Effective shear cross section $C = \frac{5}{6} F = 10$ in^2

ν_n [Hz]	Analytical solution		Matrix displacement method Number of elements			
	without shear def. and rotational inertia	with	4	6	8	10
ν_1	345.81	333.5	333.9	333.7	333.6	333.5
ν_2	1388.2	1216.7	1241.2	1226.9	1222.3	1220.3
ν_3	3112.3	2434.1	2615.9	2513.5	2478.0	2462.0
ν_4	5532.9	3825.6	5386.1	4121.6	3992.8	3932.2
ν_5	8645.2	5301.8		6003.4	5736.5	5582.8
ν_6	12449.1	6816.8		10676.2	7689.9	7405.1
ν_7	16945.6	8347.3			9681.5	9389.5
ν_8	22131.7	9881.2			11606.9	11261.6

TABLE 6.5. TRANSFORMATION MATRIX a_N WITH BIAXIAL FLEXURE AND TORSION

/200

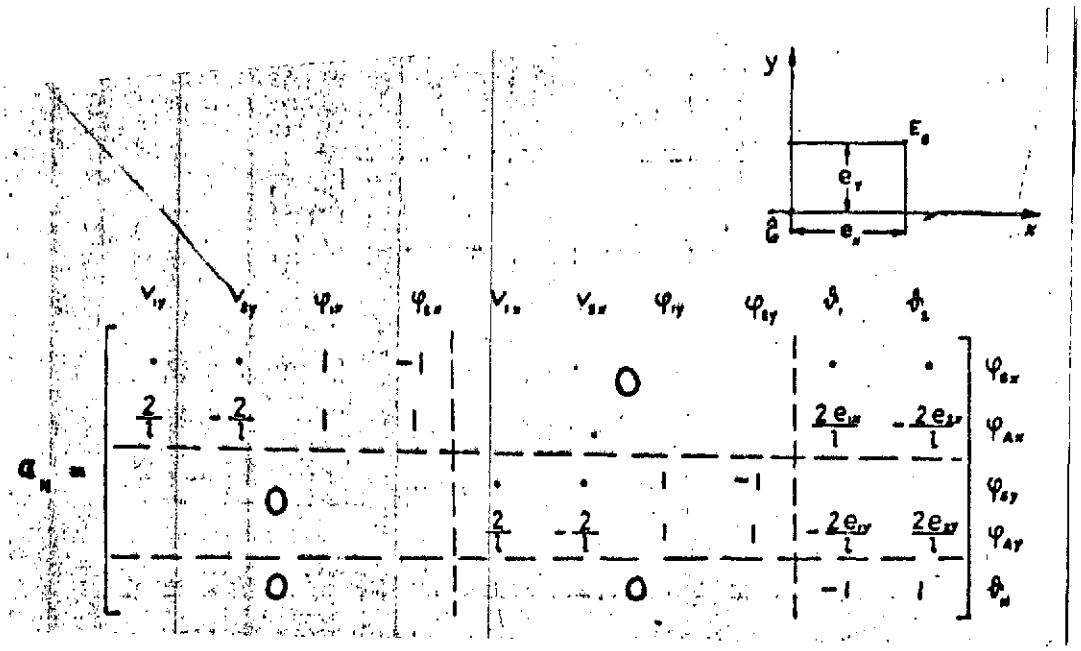

TABLE 6.6. NATURAL ELASTIC COMPLIANCES OF TWISTED BLADE ELEMENT FOR FLEXURAL AND SHEAR LOADING REFERRED TO DIRECTIONS OF PRINCIPAL AXES AT CENTER OF ELEMENT

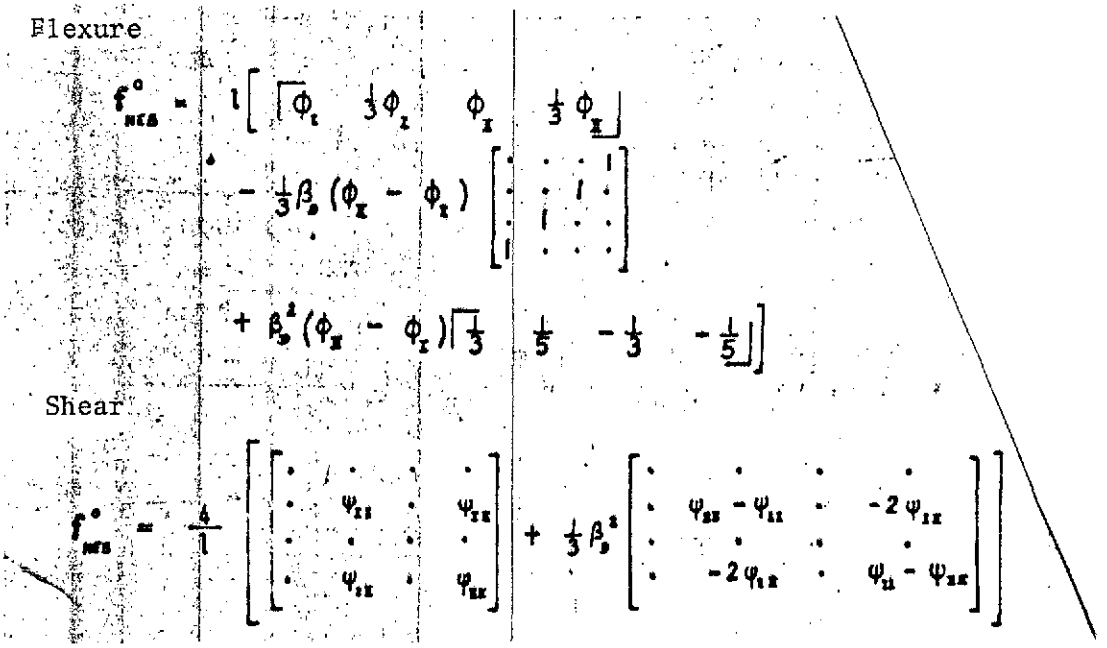

TABLE 7.1. MATRIX b OF SECTION MOMENTS AT NODES DUE TO UNIT LOADS AND MOMENTS

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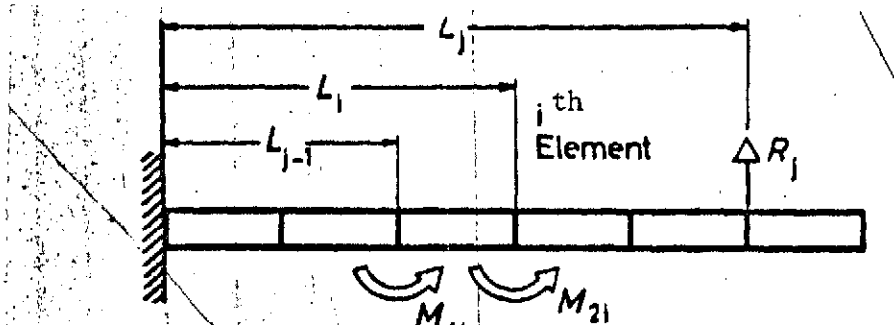
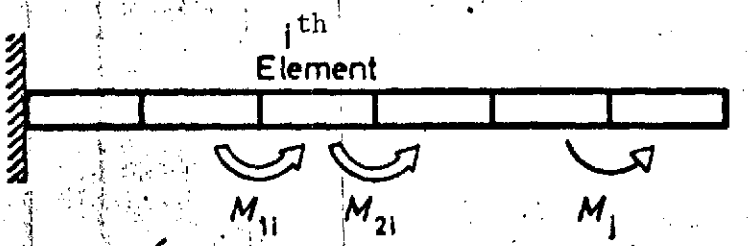


$M_{1i} = (L_j - L_{i-1})R_j$ $\quad M_{2i} = (L_j - L_i)R_j$ for $i \le j$

$M_{1i} = M_{2i} = 0$ for $i > j$



$$M_{1i} = M_{2i} = \begin{cases} M_j & \text{for } i \le j \\ 0 & \text{for } i > j \end{cases}$$

Due to: R_1 R_2 R_3 ..... in Element:

$$b_R = \begin{bmatrix} L_1 & L_2 & L_3 & \cdots & 1 \\ \bullet & L_2-L_1 & L_3-L_1 & \cdots & \\ \bullet & L_2-L_1 & L_3-L_1 & \cdots & 2 \\ \bullet & \bullet & L_3-L_2 & \cdots & \\ \bullet & \bullet & L_3-L_2 & \cdots & 3 \\ \bullet & \bullet & \bullet & \cdots & \vdots \end{bmatrix}$$

$(2s \times s)$

M_0 M_1 M_2 M_3 ....

$$b_M = \begin{bmatrix} \bullet & 1 & 1 & 1 & \cdots & 1 \\ \bullet & 1 & 1 & 1 & \cdots & \\ \bullet & \bullet & 1 & 1 & \cdots & 2 \\ \bullet & \bullet & 1 & 1 & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \end{bmatrix}$$

$(2s \times s+1)$

$$b = [b_R \quad b_M] \quad (2s \times 2s+1)$$

TABLE 72. TOTAL BUILDUP OF MASS MATRIX M

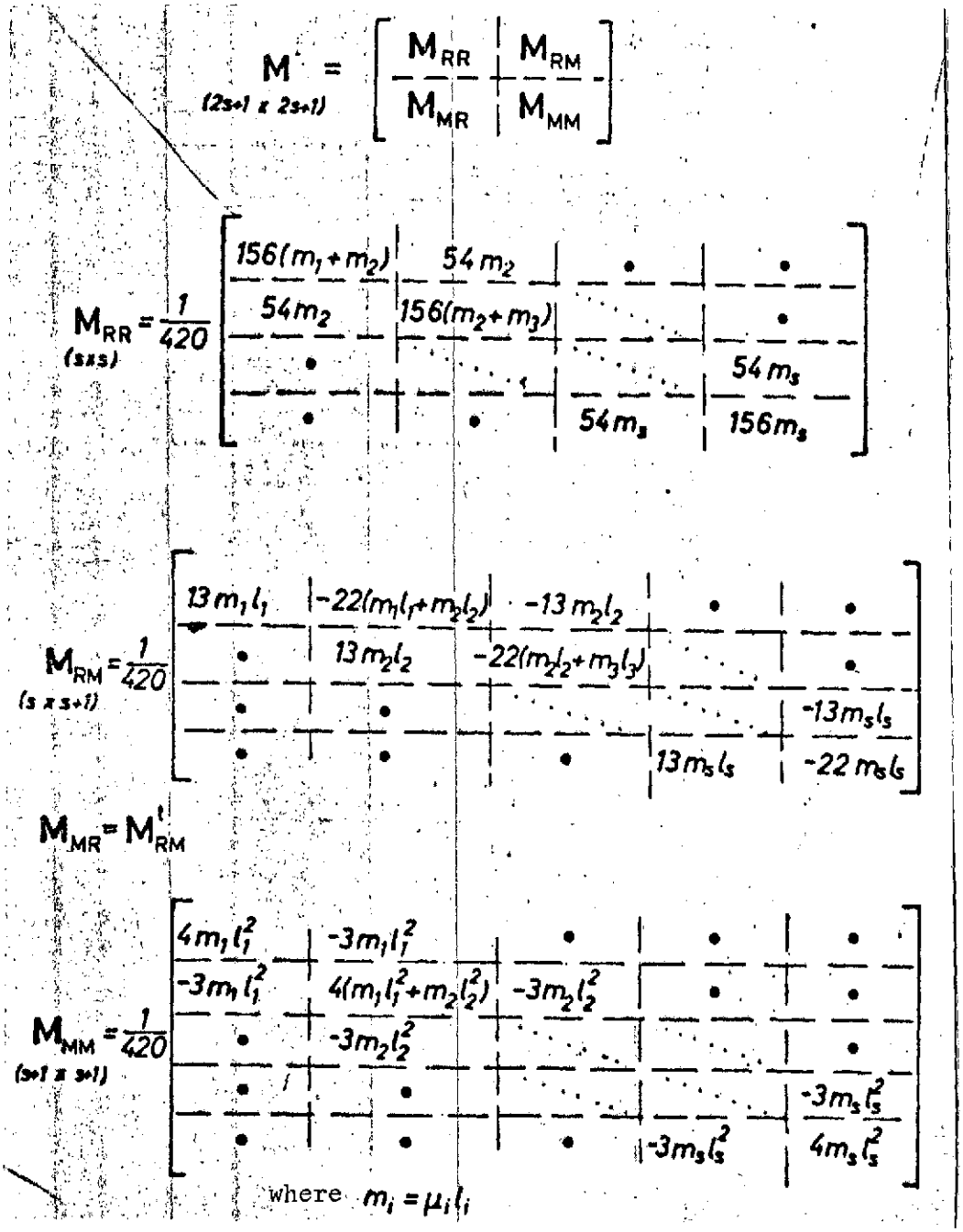
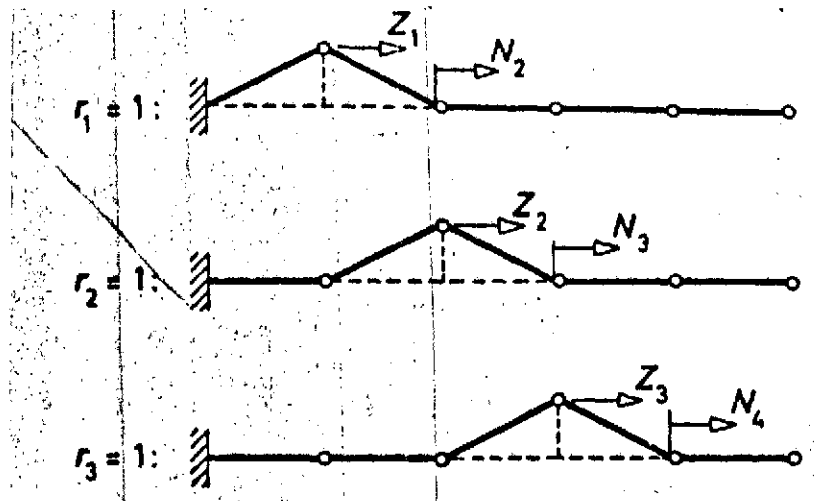

M = [M_RR | M_RM ; M_MR | M_MM]
(2s+1 x 2s+1)
M_RR = 1/420 [156(m1+m2) 54m2 . . ; 54m2 156(m2+m3) . . ; . . 54ms ; . . 54ms 156ms]
(sxs)
M_RM = 1/420 [13m1l1 -22(m1l1+m2l2) -13m2l2 . . ; . 13m2l2 -22(m2l2+m3l3) . . ; . . . -13msls ; . . . 13msls -22msls]
(s x s+1)
M_MR = M'_RM
M_MM = 1/420 [4m1l1^2 -3m1l1^2 . . . ; -3m1l1^2 4(m1l1^2+m2l2^2) -3m2l2^2 . . ; . -3m2l2^2 . . . ; . . . -3msls^2 ; . . . -3msls^2 4msls^2]
(s+1 x s+1)
where mi = μi li

TABLE 7.3. MATRIX C_r OF SECTION MOMENTS AT NODES DUE TO UNIT DISPLACEMENT $r_i = 1$ /203



Node : 0 1 2 3 . . . . s

Element : 1 2 3 . . . . s

Due to: r_1 r_2 r_3 .... r_s in Element:

C_r = ($2s \times s$)

Z_1	Z_2	Z_3		Z_s	1
$-N_2$	Z_2	Z_3		Z_s	
$-N_2$	Z_2	Z_3		Z_s	2
•	$-N_3$	Z_3		Z_s	
•	$-N_3$	Z_3		Z_s	3
•	•	$-N_4$		Z_s	
•	•	$-N_4$		Z_s	⋮
. . . . .					
•	•	•		Z_s	s
•	•	•		•	
•					

TABLE 7.4.. MATRIX C_ϕ OF NORMAL FORCES IN ELEMENTS DEFORMED BY $\phi_i = 1$ /204

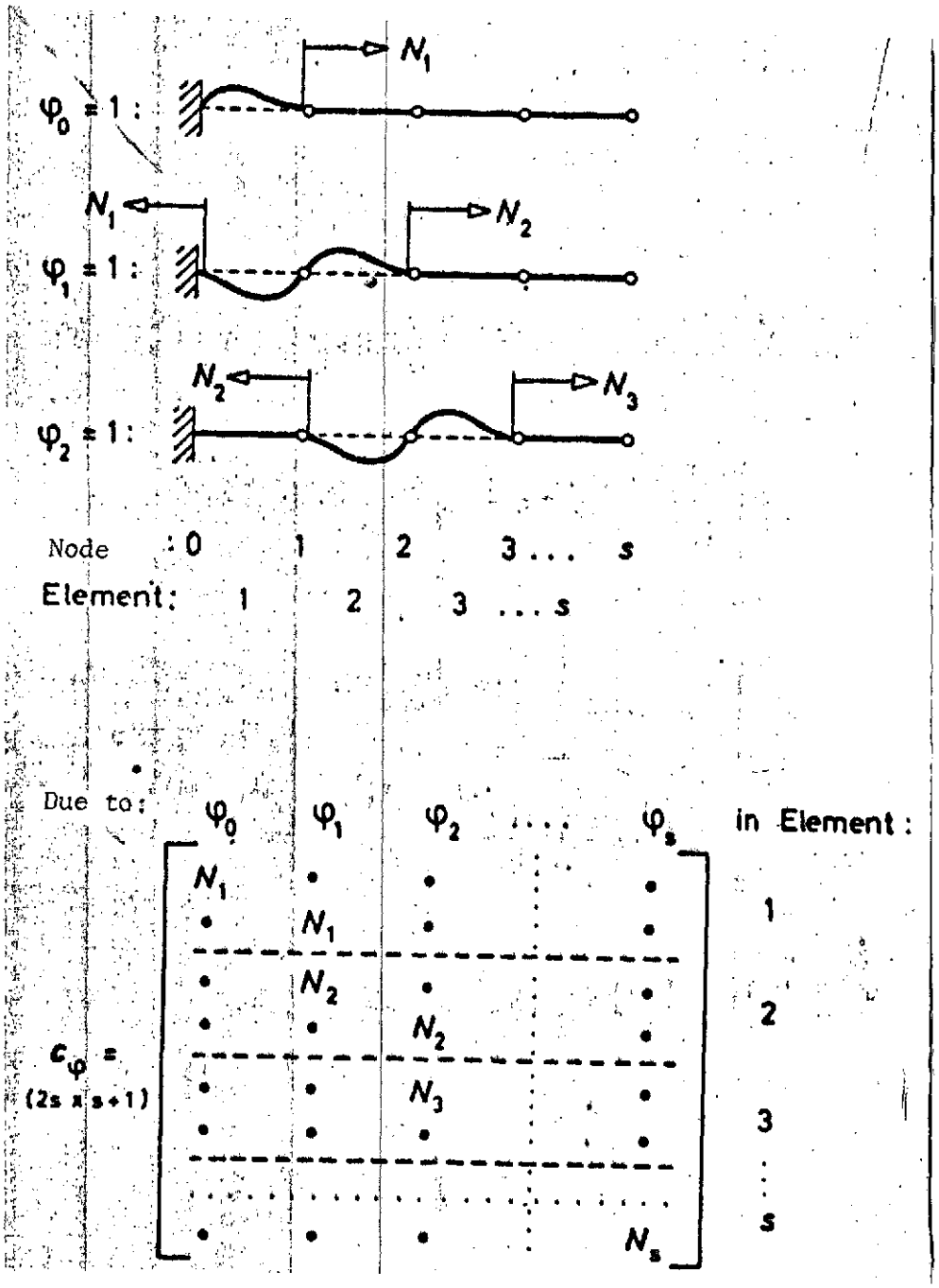


TABLE 7.5. DERIVATION OF EXPANDED COMPLIANCE MATRIX FOR A BEAM ELEMENT TO TAKE NORMAL FORCES INTO CONSIDERATION WHEN ELASTIC CURVES ARE THIRD-ORDER PARABOLAS

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$$f_{eij} = \int_0^l m_i \frac{1}{EI} M_j \, dz'$$

where $M = N r_{\zeta'}$ or $M = N(1 - r_{\zeta'})$

$z' = l\zeta'$ $\quad dz' = l \, d\zeta'$

$r_{\zeta'}$ \ $M_{\zeta'}$ / $m_{\zeta'}$	M_0	M_1	N	N
	$r_{\zeta'} = \zeta^2(3-2\zeta')r_1$ $r_{\zeta'} = (1-\zeta')^2(1+2\zeta')r_0$	$r_{\zeta'} = (1-\zeta')^2(1+2\zeta')r_0$ $r_{\zeta'} = \zeta^2(3-2\zeta')r_1$	$r_{\zeta'} = (1-\zeta')^2 \zeta \varphi_0 l$	$r_{\zeta'} = -\zeta^2(1-\zeta')\varphi_1 l$
	$(1-\zeta')^2(1+2\zeta')M_0$	$\zeta^2(3-2\zeta')M_1$	$-(1-\zeta')^2\zeta\varphi_0 lN$	$\zeta^2(1-\zeta')\varphi_1 lN$
$1 - \zeta'$	$\frac{21}{60}\frac{l}{EI}M_0$	$\frac{9}{60}\frac{l}{EI}M_1$	$-\frac{3}{60}\frac{l^2}{EI}\varphi_0 N$	$\frac{2}{60}\frac{l^2}{EI}\varphi_1 N$
ζ'	$\frac{9}{60}\frac{l}{EI}M_0$	$\frac{21}{60}\frac{l}{EI}M_1$	$-\frac{2}{60}\frac{l^2}{EI}\varphi_0 N$	$\frac{3}{60}\frac{l^2}{EI}\varphi_1 N$

TABLE 7.6. NATURAL FREQUENCIES OF A HOMOGENEOUS FLEXIBLE BEAM UNDER CONSTANT NORMAL FORCE; COMPARISON WITH ANALYTICAL SOLUTION AND EXPERIMENT

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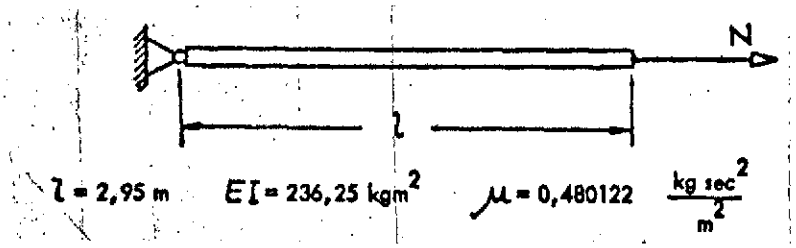

First line — Matrix method 20 elements
Second line — Differential equation
Third line — Experimental

ν [Hz] \ N [kp]	0	1000	2000	3000	4000
ν_1	-	3,953	5,536	6,757	7,789
	-	3,944	5,520	6,73	7,749
	-	-	-	-	-
ν_2	6,252	13,79	18,21	21,67	24,618
	6,255	13,809	18,206	21,665	24,617
	6,283	-	-	-	-
ν_3	20,26	28,84	35,16	40,38	44,94
	20,270	28,854	35,160	40,369	44,913
	20,43	-	-	-	-
ν_4	42,30	51,05	58,40	64,72	70,53
	42,292	50,996	58,285	64,650	70,367
	42,63	-	-	-	-
ν_5	72,41	81,10	88,82	95,93	102,56
	72,321	80,935	88,655	95,688	102,180
	72,53	-	-	-	-
ν_6	110,58	119,10	127,16	134,65	141,73
	110,358	118,846	126,750	134,145	141,116
	111,0	-	-	-	-

[Note: Commas in numerals are equivalent to decimal points.]

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